Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 27, 2013

METHOD OF A SMALL PARAMETER FOR THE NONLINEAR THEORY OF NON-SHALLOW SHELLS

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Abstract. In this present paper is suggested the method of a small parameter for I. Vekua's and Koiter-Naghdi's non-shallow shells.

Keywords and phrases: Non-shallow shells, metric tensor and tensor of curvature, midsurface of the shell.

AMS subject classification: 74K25, 74B20.

1. Non-Shallow and Shallow Shells

Let $\mathbf{R}$ be a radius-vector a point with coordinates (x^1, x^2, x^3), that is

$$\mathbf{R}(x^1,x^2,x^3) = \mathbf{r}(x^1,x^2) + x^3\mathbf{n}(x^1,x^2),$$

where $\mathbf{r}$ and $\mathbf{n}$ are, respectively, radius-vector and unit vector of the normal to the midsurface S at the (x^1, x^2). If $2h$ is the shell thickness, then $-h \leq x_3 \leq h$.

The so-called local basis vectors have the form:

a) For non-shallow shells [1]:

$$\begin{aligned}&\partial_\alpha\mathbf{R} = \mathbf{R}_\alpha = (a_\alpha^\beta - x_3 b_\alpha^\beta)\mathbf{r}_\beta, \quad \mathbf{R}^\alpha = \vartheta^{-1}(a_\beta^\alpha + x_3 L_\beta^\alpha)\mathbf{r}_\beta,\\ &\partial_3\mathbf{R} = \mathbf{R}_3 = \mathbf{R}^3 = \mathbf{n}, \quad (\alpha,\beta = 1,2; x_3 = x^3)\end{aligned} \tag{1}$$

where $a_\alpha^\beta(a_{\alpha\beta}, a^{\alpha\beta})$ and $b_\alpha^\beta(b_{\alpha\beta}, b^{\alpha\beta})$ are components, respectively, of the metric and curvatures tensors of surface S, H and K are middle and principal (Gaussian) curvatures of the surface S,

$$2H = b_1^1 + b_2^2, \quad K = b_1^1 b_2^2 + b_2^1 b_1^2, \quad \vartheta = 1 - 2Hx_3 + Kx_3^2, \quad L_\beta^\alpha = b_\beta^\alpha - 2Ha_\beta^\alpha.$$

b) For non-shallow shells (Koiter-Naghdi) we have

$$\mathbf{R}_\alpha = (a_\alpha^\beta - x_3 b_\alpha^\beta)\mathbf{r}_\beta, \quad \mathbf{R}^\alpha = (a_\beta^\alpha + x_3 b_\beta^\alpha)\mathbf{r}^\beta, \quad \mathbf{R}_3 = \mathbf{R}^3. \tag{2}$$

c) For shallow shells it may be assumed that

$$\mathbf{R}_\alpha \cong \mathbf{r}_\alpha, \quad \mathbf{R}^\alpha \cong \mathbf{r}^\alpha \quad \Rightarrow \quad x_3 b_\alpha^\beta \cong 0 \quad \Rightarrow \quad h b_\alpha^\beta \cong 0. \tag{3}$$

2. Equations of equilibrium an elastic medium (vector and tensor notes)

$$\begin{aligned}&\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}\boldsymbol{\sigma}^i}{\partial x^i} + \boldsymbol{\Psi} = 0, \quad (g = det\{g_{ij}\}, \quad g_{ij} = \mathbf{R}_i\mathbf{R}_j, \quad i,j = 1,2,3),\\ &\boldsymbol{\sigma}^i = E^{ijpq}e_{pq}(\mathbf{R}_j + \partial_j\mathbf{U}), \quad (geometrically\ nonlinear),\\ &E^{ijpq} = \lambda g^{ij}g^{pq} + \mu(g^{ip}g^{jq} + g^{iq}g^{jp}), \quad (g_{ij} = \mathbf{R}^i\mathbf{R}^j),\\ &e_{pq} = \frac{1}{2}(\mathbf{R}_q\partial_p\mathbf{U} + \mathbf{R}_p\partial_q\mathbf{U} + \partial_p\mathbf{U}\partial_q\mathbf{U}), \quad (p,q = 1,2,3),\end{aligned} \tag{4}$$

where $\boldsymbol{\sigma}^i = \sigma^{ij}(\mathbf{R}_j + \partial_j \mathbf{U})$ are contravariant constituents of the stress tensor, $\mathbf{U}$ is the displacement vector. Since $g = a\vartheta^2$ $(a = det\{a_{\alpha\beta}\})$ and $\vartheta\boldsymbol{\sigma}^i = \mathbf{T}^i$, then equations (4) for elastic shells can be rewritten as

$$\nabla_\alpha \mathbf{T}^\alpha + \partial_3 \mathbf{T}^3 + \mathbf{\Phi} = 0, \;\; (\mathbf{\Phi} = \vartheta \mathbf{\Psi}) \;\Rightarrow$$

$$\begin{array}{l} \nabla_\alpha T^{\alpha\beta} - b_\alpha^\beta T^{\alpha 3} + \Phi^\beta = 0, \\ \nabla_\alpha T^{\alpha 3} + b_{\alpha\beta} T^{\alpha 3} + \Phi^3 = 0, \end{array} \tag{5}$$

where ∇_α are covariant derivatives with respect to the x^1, x^2 - Gaussian parameters of the surface S, $T^{ij} = \mathbf{T}^i \cdot \mathbf{r}^j$ - contravariant components of the stress tensor, $\Phi^i = \vartheta\Psi^i$, further

$$\begin{array}{r} \mathbf{T}^i = \vartheta\boldsymbol{\sigma} = \dfrac{1}{2}\vartheta A_{i_1}^i M^{i_1 j_1 p_1 q_1}[A_{p_1}^p(\mathbf{r}_{q_1}\partial_p \mathbf{U}) + A_{q_1}^q(\mathbf{r}_{p_1}\partial_q \mathbf{U}) + A_{p_1}^p A_{q_1}^q \partial_p \mathbf{U} \partial_q \mathbf{U}] \\ \times(\mathbf{r}_{j_1} + A_{j_1}^j \partial_j \mathbf{U}) \end{array} \tag{6}$$

$$\begin{array}{l} A_{i_1}^i = \mathbf{R}^i \mathbf{r}_{i_1} \;\Rightarrow\; A_{\alpha_1}^\alpha = \vartheta^{-1}[a_{\alpha_1}^\alpha + x_3(b_{\alpha_1}^\alpha - 2H a_{\alpha_1}^\alpha)], \\ A_3^\alpha = A_\alpha^3 = 0, \;\; A_3^3 = \mathbf{nn} = 1, \\ M^{i_1 j_1 p_1 q_1} = \lambda a^{i_1 j_1} a^{p_1 q_1} + \mu(a^{i_1 p_1} a^{j_1 q_1} + a^{i_1 q_1} a^{j_1 p_1}), \\ (a^{\alpha\beta=} = \mathbf{r}^\alpha \cdot \mathbf{r}^\beta, \;\; a^{\alpha 3} = 0, \;\; a^{33} = 1). \end{array} \tag{7}$$

Then by means of I. Vekua's method 3-D problems are reduced to the 2-D problems of the theory shells:

$$\int_{-h}^{h} (\nabla_\alpha \mathbf{T}^\alpha + \partial_3 \mathbf{T}^3 + \mathbf{\Phi}) P_m\left(\frac{x_3}{h}\right) dx_3 = 0 \;\;\Rightarrow$$

$$(m = 0, \; 1, ...)$$

$$\begin{array}{l} \nabla_\alpha \overset{(m)}{T}{}^{\alpha\beta} - b_\alpha^\beta \overset{(m)}{T}{}^{\alpha 3} - \dfrac{2m+1}{h}\left(\overset{(m-1)}{T}{}^{3\beta} + \overset{(m-3)}{T}{}^{3\beta} + \cdots\right) + \overset{(m)}{F}{}^\beta = 0, \\ \nabla_\alpha \overset{(m)}{T}{}^{\alpha 3} + b_{\alpha\beta} \overset{(m)}{T}{}^{\alpha\beta} - \dfrac{2m+1}{h}\left(\overset{(m-1)}{T}{}_3^3 + \overset{(m-3)}{T}{}_3^3 + \cdots\right) + \overset{(m)}{F}{}^3 = 0, \end{array} \tag{8}$$

where $P_m\left(\frac{x_3}{h}\right)$ is Legendre polynomials in the interval $x_3 \in [-h, \; h]$,

$$\left(\overset{(m)}{\mathbf{T}}{}^i, \; \overset{(m)}{\mathbf{U}}, \; \overset{(m)}{\mathbf{\Phi}}, \right) = \frac{2m+1}{h}\int_{-h}^{h}\left(\mathbf{T}^i, \; \mathbf{U}, \; \mathbf{\Phi}\right) P_m\left(\frac{x_3}{h}\right) dx_3,$$

$$\overset{(m)}{\boldsymbol{F}} = \overset{(m)}{\mathbf{\Phi}} + \frac{2m+1}{h}\left(\overset{(+)}{\mathbf{T}}{}^3 - (-1)^m \overset{(-)}{\mathbf{T}}{}^3\right), \quad \overset{(\pm)}{\mathbf{T}}{}^3 = \mathbf{T}^3\left(x^1, x^2, \pm h\right).$$

Now we have the following integrals

a) for shallow shells (see 1-3):

$$\int_{-h}^{h} P_s P_m dx_3 = \frac{2h}{m+s+1}\delta_{ms}, \quad \int_{-h}^{h} P_{s_1} P_{s_2} P_m dx_3, \quad \int_{-h}^{h} P_{s_1} P_{s_2} P_{s_3} P_m dx_3.$$

It should be noted that those integrals can be calculation by means Adams formulas

$$P_m(x)P_n(x) = \sum_{r=0}^{\min(m,n)} \frac{A_{m-r}A_rA_{n-r}}{A_{m+n-r}} \frac{2(m+n)-4r+1}{2(m+n)-2r+1} P_{m+n-2r}(x),$$

$$A_m = \frac{1\cdot 3\cdots(2m-1)}{m!}.$$

b) For non-shallow shells (Koiter-Naghdi) we have the following integrals:

$$\int_{-h}^{h} x_3^k P_{s_1}\cdots P_{s_n} P_m dx_3, \quad (n=1,2,3;\ k=0,1,...,5),$$

which calculation also by Adams formulas.

c) For non-shallow shells (I. Vekua) we have integrals of the types

$$\int_{-h}^{h} \frac{x_3^k P_{s_1}\cdots P_{s_n} P_m}{(1-2Hx_3+Kx_3^2)^n} dx_3, \quad (n=0,1,2,3;\ k=0,1,...,4),$$

which calculation by Adams and F. Neuman formulas:

$$\frac{1}{2}\int_{-1}^{1} \frac{P_m(y)dy}{x-y} = Q_m(x), \quad |x|>1 \ \ (F.Neuman),$$

where $Q_m(x)$ is the Legendre function on the second kind.

For example we have

$$\int_{-h}^{h} \frac{(a_{\alpha_1}^{\alpha} - x_3 L_{\alpha_1}^{\alpha})(a_{\beta_1}^{\beta} - x_3 L_{\beta_1}^{\beta})}{1-2Hx_3+Kx_3^2} P_s P_m dx_3$$

$$= \left[\frac{B_{\alpha_1}^{\alpha}(hy)B_{\beta_1}^{\beta}(hy)}{2\sqrt{E}h} \begin{pmatrix} P_m(y)Q_s(y), & m\le s \\ Q_m(y)P_s(y), & m\ge s \end{pmatrix}\right]_{y_1}^{y_2} + \frac{2h}{m+s+1}\frac{L_{\alpha_1}^{\alpha}L_{\beta_1}^{\beta}}{K}\delta_{ms},$$

where $B_{\alpha_1}^{\alpha}(hy) = a_{\alpha_1}^{\alpha} + hyL_{\alpha_1}^{\alpha}$, $L_{\alpha_1}^{\alpha} = b_{\alpha_1}^{\alpha} - 2Ha_{\alpha_1}^{\alpha}$, $E = H^2 - K$,

$$[f(y)]_{y_1}^{y_2} = f(y_2) - f(y_1), \quad y_{1,2} = \left[(H \mp \sqrt{E})h\right]^{-1}.$$

3. Introduction of a small parameter

Let $\rho = \max\{b_{\beta}^{\alpha},\ b_{\alpha\beta},\ b^{\alpha\beta}\} \Rightarrow h<\rho \Rightarrow \varepsilon = \frac{h}{\rho} < 1$ $(x^1, x^2 \in S)$.

Therefore they can be represented as follows

$$|\varepsilon b_{\alpha}^{\beta}\varrho| \le q < 1,$$

where ε is a small parameter, h is semithickness of shell.

Now we assume the validity of the expansions for the approximation of order N:

$$\left(\overset{(m)}{\mathbf{T}^i},\ \overset{(m)}{\mathbf{U}},\ \overset{(m)}{\boldsymbol{F}},\right)=\sum_{n=1}^{\infty}\left(\overset{(m,n)}{\mathbf{T}^i},\ \overset{(m,n)}{\mathbf{U}},\ \overset{(m,n)}{\boldsymbol{F}},\right)\varepsilon^n,\quad (m=0,1,...,N).$$

Substituting the above expansions into the relations (8) and equalizing the coefficients of expansions for ε^n, we obtain the following 2-D finite system of equilibrium equations with respect to the components of displacement vector in the isometric coordinates:

$$\begin{aligned}&4\mu\partial_{\overline{z}}\Big(\Lambda^{-1}\partial_z\overset{(m,n)}{u_+}\Big)+2(\lambda+\mu)\partial_{\overline{z}}\overset{(m,n)}{\theta}+\frac{2\lambda}{h}\partial_{\overline{z}}\overset{(m,n)}{u_3'}-\frac{2m+1}{h}\mu\Big[2\partial_{\overline{z}}\Big(\overset{(m-1,n)}{u_3}+\\&\overset{(m-3,n)}{u_3}+\cdots\Big)+\overset{(m-1,n)}{u_+'}+\overset{(m-3,n)}{u_+'}+\cdots\Big]+\overset{(m,n)}{F_+}=0,\\&\mu\Big(\nabla^2\overset{(m,n)}{u_3}+\overset{(m,n)}{\theta'}\Big)-\frac{2m+1}{h}\Big[\lambda\Big(\overset{(m-1,n)}{\theta}+\overset{(m-3,n)}{\theta}+\cdots\Big)+\\&(\lambda+2\mu)\Big(\overset{(m-1,n)}{u_3'}+\overset{(m-3,n)}{u_3'}+\cdots\Big)\Big]+\overset{(m,n)}{F_3}=0,\end{aligned}\tag{9}$$

where $u_+=u_1+iu_2$, $\theta=\Lambda^{-1}\Big(\partial_z u_++\partial_{\overline{z}}\overline{u}_+\Big)$, $\overset{(m)}{u'}=\frac{2m+1}{h}\Big(\overset{(m+1)}{u}+\overset{(m+3)}{u}+\cdots\Big)$, $ds^2=\Lambda(z,\overline{z})dzd\overline{z}$, $z=x^1+ix^2$,$2\partial_z=\partial_1-i\partial_2$, $\nabla^2=\frac{4}{\Lambda}\frac{\partial^2}{\partial z\partial\overline{z}}$.

Obviously, in passing from the n-th step to the $(n+1)$-th step only the right hand of equations are changed.

Note that the system (9) we can write as

$$\begin{aligned}&\mu\Delta\overset{(m)}{u_+}+2(\lambda+\mu)\partial\overline{z}\overset{(m)}{\theta}+\overset{(m)}{L_+}\left(\overset{(1)}{u_i},...,\overset{(N-1)}{u_i}\right)=0,\quad (L_+=L_1+iL_2)\\&\mu\Delta\overset{(m)}{u_3}+\overset{(m)}{L_3}\left(\overset{(1)}{u_i},...,\overset{(N-1)}{u_i}\right)=0,\quad (m=0,1,...,N)\end{aligned}\tag{10}$$

where Δ is Laplace operator $\Delta=4\frac{\partial^2}{\partial z\partial\overline{z}}$ and $\overset{(m)}{L_i}$ are linear differential operators containing unknown vector-functions $\overset{(k)}{\mathbf{U}}$ and their first-order derivatives with respect to z. It is important to note that the finite system (10) makes the above suggested version of shell theory closer to the equations of the classical plane theory of elasticity to the Poisson equation.

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Received 25.05.2013; revised 30.09.2013; accepted 12.11.2013.

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