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ON ONE NONLINEAR INTEGRO-DIFFERENTIAL EQUATION WITH SOURCE TERM

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Abstract. One nonlinear integro-differential equation with source terms is considered. The model arises at describing penetration of a magnetic field into a substance and is based on Maxwell's system. Existence, uniqueness and large time behavior of solutions of the initial-boundary value problem as well as semi-discrete scheme is studied. More wide class of nonlinearity is considered than one has been already investigated in construction of semi-discrete analogue.

Keywords and phrases: Nonlinear integro-differential equation, existence, uniqueness, asymptotic behavior, semi-discrete difference scheme.

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One kind of nonlinear integro-differential model arises on mathematical simulation of the process of penetration of a magnetic field into a substance [1]. This model were introduced after reduction the nonlinear Maxwell's differential system [2] to the integro-differential form. In [3] some generalization of such type models is given. One-dimensional simple analog describing the same physical process has the following form

$$\frac{\partial U}{\partial t} - a\left(\int_0^t\int_0^1\left(\frac{\partial U}{\partial x}\right)^2 dx d\tau\right)\frac{\partial^2 U}{\partial x^2} = 0, \tag{1}$$

where $a = a(S) \geq Const > 0$ is a given function of its argument.

Many works are dedicated to the investigation and numerical resolution of the integro-differential models described in [1] and [3]. Particularly, in [1], [3]-[10] solvability and uniqueness of the initial-boundary value problems for these type equations are studied. Asymptotic behavior of solutions as $t \to \infty$ is investigated in many works also (see, for example, [9],[11]-[20] and references therein). Numerical resolution is given in works [10], [15]-[22] and in a number of other works as well.

The aim of this note is to study asymptotic behavior of solution as $t \to \infty$ and investigate convergence of corresponding semi-discrete scheme for one generalization of the equation type (1) by adding monotonic nonlinear term. Existence and uniqueness of solutions of the initial-boundary value problem is studied as well. More wide class of nonlinearity is considered than one has been already investigated in construction of semi-discrete analogue. The investigated equation has the form

$$\frac{\partial U}{\partial t} - \left(1+\int_0^t\int_0^1\left(\frac{\partial U}{\partial x}\right)^2 dx d\tau\right)^p\frac{\partial^2 U}{\partial x^2} + |U|^{q-2}U = 0, \tag{2}$$

where $0 < p \le 1$ and $q \ge 2$.

Let us note that such kind generalizations for the equation described in [1] is made in [18] and for (2) type equation is discussed in the work [20].

In the $[0,1] \times [0,\infty)$ let us consider following initial-boundary value problem:

$$\begin{gathered} U(0,t) = U(1,t) = 0, \\ U(x,0) = U_0(x), \end{gathered} \tag{3}$$

where $U_0 = U_0(x)$ is given function.

The following statement of existence, uniqueness and asymptotic behavior of the solution is true.

Theorem 1. *If $0 < p \le 1$, $q \ge 2$ and $U_0 \in H_0^1(0,1)$, then where exist unique solution of problem (2),(3) and the following asymptotic property takes place*

$$\|U\| + \left\| \frac{\partial U}{\partial x} \right\| \le C \exp\left(-\frac{t}{2}\right).$$

Here $\|\cdot\|$ is the usual norm of the space $L_2(0,1)$ and C denotes positive constant independent of t.

On [0,1] let us introduce a net with mesh points denoted by $x_i = ih$, $i = 0, 1, \ldots, M$, with $h = 1/M$. The boundaries are specified by $i = 0$ and $i = M$. The semi-discrete approximation at (x_i, t) is designed by $u_i = u_i(t)$. The exact solution to the problem at (x_i, t) is denoted by $U_i = U_i(t)$. At points $i = 1, 2, \ldots, M-1$, the integro-differential equation will be replaced by approximation of the space derivatives by a forward and backward differences. We will use the following known notations:

$$r_{x,i}(t) = \frac{r_{i+1}(t) - r_i(t)}{h}, \quad r_{\bar{x},i}(t) = \frac{r_i(t) - r_{i-1}(t)}{h}, \quad r_{\bar{x}x,i}(t) = \frac{r_{i+1}(t) - 2r_i(t) + r_{i-1}(t)}{h^2}.$$

Let us correspond to problem (2),(3) the following semi-discrete scheme:

$$\begin{gathered} \frac{du_i}{dt} = \left(1 + h \sum_{i=1}^{M} \int_0^t (u_{\bar{x},i})^2 \, d\tau \right)^p u_{\bar{x}x,i} - |u_i|^{q-2} u_i, \\ i = 1, 2, \ldots, M-1, \end{gathered} \tag{4}$$

$$u_0(t) = u_M(t) = 0, \tag{5}$$

$$u_i(0) = U_{0,i}, \quad i = 0, 1, \ldots, M. \tag{6}$$

So, we obtained Cauchy problem (4)-(6) for nonlinear system of ordinary integro-differential equations.

Introduce inner product and norm:

$$(r, g) = h \sum_{i=1}^{M-1} r_i g_i, \quad \|r\| = (r, r)^{1/2}.$$

The following statement takes place.

Theorem 2. *If $0 < p \leq 1$, $q \geq 2$ and problem (2),(3) has a sufficiently smooth solution $U = U(x,t)$, then the solution $u = u(t) = (u_1(t), u_2(t), \ldots, u_{M-1}(t))$ of problem (4)-(6) tends to $U = U(t) = (U_1(t), U_2(t), \ldots, U_{M-1}(t))$ as $h \to 0$ and the following estimate is true*

$$\|u(t) - U(t)\| \leq Ch.$$

Note that investigated semi-discrete scheme (4) - (6) is using for numerical solution of the problem (2), (3) by natural discretisation of time derivative and integral as it is given for example in [22] for the case $p = 1$. Solving the obtaining finite difference scheme we use a algorithm analogical to [17]. So, it is necessary to use Newton iterative process. According to this method the great numbers of numerical experiments are carried out. These experiments agree with the theoretical results given in the Theorems 1 and 2.

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