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QUANTUM EVENTS AND TENSOR PROBABILITIES

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Abstract. The notion of quantum events will be done. The probability of these events be found non constant but having tensor character. A geometric interpretation of statistical application will be done.

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1. Introduction. In this paper we treat with questions which are well known in a statistical approach. For the distribution function to estimate based on a empirical distribution, we demand the sample space to be ordered in any way - often Euclidean space is taken. Generally, distribution function is non scale invariant; grouping the data in different ways, an empirical distribution will be varied, that follows change of theoretical distribution function. It means a probabilistic model performance. Which of many existing models is preferable is difficult to decide. The arising problem to solve, we deal with many distributions and find a law, managed with diverse models, that is having covariant character. In consequence of these the notions of quantum events and tensor probabilities are introduced.

2. Definitions. Let (Ω, F, P) be a classical probability space and $\{A_i\}$, $i = 1, 2, ..., n$ be any complete system of events. For any $C \in F$ the number

$$P(C) = \sum_{i=1}^{n} P(A_i) P_{A_i}(C)$$

is well defined. If $C \notin F$ but $CA_i \in F$, for all $i = 1, 2, ..., n$, C will be named as *quantum event* and characterized by systems $\{P(A_i)\}$ and $\{P_{A_i}(C)\}$ having the tensor characters. If $\{B_j\}$, $j = 1, 2, ..., m$ is other complete system, we have

$$P(A_i) = \sum_{j=1}^{m} P(B_j) P_{B_j}(A_i)$$

and $\{A_i\}$ presents *contra-variant tensor*. By the formula

$$P(C) = \sum_{i=1}^{n} P(A_i) P_{A_i}(C)$$

and private principle $P_{A_i}(C)$ has tensor character - we respect it as *covariant tensor*. They are 1-rank tensors. The conditional probabilities $P_{B_j}(A_i)$ present *mixed tensor* not of the 1-rank, but 2-rank tensor in view its freedom of indexes.

For become to higher rank tensors we use the inclusion-exclusion formula. Let $\{A_i\},\ \ i=1,2,...,n$ be non disjoint but system with the property $\bigcup A_i=\Omega$. We have

$$P(C)=\sum P(A_i)P_i(C);$$

$$P_i(C)=P_{A_i}(C)-\sum P_{A_i}(A_j)P_{A_iA_j}(C)+\sum P_{A_iA_j}(A_k)P_{A_iA_jA_k}(C)-\cdots.$$

$$P_i(C)=P_{A_i}(C)-\sum P_{A_i}(A_j)P_{ij}(C);$$

$$P_{ij}(C)=P_{A_iA_j}(C)-\sum P_{A_iA_j}(A_k)P_{A_iA_jA_k}(C)+\cdots.$$

$$P_{ij}(C)=P_{A_iA_j}(C)-\sum P_{A_iA_j}(A_k)P_{ijk}(C);$$

$$P_{ijk}(C)=P_{A_iA_jA_k}(C)-\cdots,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$P_{i_1,i_2,...,i_n}=P_{A_1A_2...A_n},$$

for all permutations $i_1,i_2,...,i_n$ of $1,2,...,n$.

Using the formulas

$$P_{A_i}(A_j)=\sum_k \Gamma^j_{ik}P(A_k),$$

$$P_{A_iA_j}(C)=\sum_k \Gamma^k_{ij}P_{A_k}(C),$$

$$T^k_{(ij)}=\Gamma^k_{ij}-\Gamma^k_{ji},$$

$$R^l_{ijk}=\sum_p(\Gamma^l_{pj}\Gamma^p_{ik}-\Gamma^l_{pi}\Gamma^p_{kj}),$$

we write

$$P_{ij}-P_{ji}=\sum_k T^k_{(ij)}P_{A_k}(C)+\sum_{k,l}R^l_{ijk}P(A_k)P_{A_l}(C).$$

$T^k_{(ij)}$ and R^l_{ijk} turn out to have tensor property. They are so called torsion and curvature tensors. If they vanishes we come to probabilistic formula for complete systems.

3. Quantum manifold. Consider a collection of quantum events M. That is, for $C\in M$ there is an σ-algebra F such that $CA_i\in F$, for $A_i\in F$, $i=1,2,...,n$ and $B_j\in F$, $j=1,2,...,m$. The probabilities of coordinate events satisfy the relation

$$P(A_i)=\sum_{j=1}^m P(B_j)P_{B_j}(A_i).$$

We are speaking that a tensor field P_{A_i} is given on the manifold, if for each point $C \in M$, the tensor $P_{A_i}(C)$ is pinned at the point $C \in M$. For two tensor fields P_{A_i} and P_{B_i} define *affine probabilistic connection* $P_{A_i}^{B_j}$ as follows -

$$P_{A_i}^{B_j}(C) = \sum_k \Gamma_{ij}^k P_{A_k}(C).$$

Starting from tensor fields writing down in coordinate system we go to general connections of tensor fields.

4. Application to statistics. Let $x = (x_1, x_2, ..., x_n) \in R_n$ be any sample. For given various data, ordered in any ways $z_j(x)$, $j = 1, 2, ..., m$, $m \leq n$, the empirical distribution $p_j(z) = \#\{x_{r_j} : x_{r_j} = z_j\}/n$ possesses a tensor character. The couples $(z, p_j) = y_j$, $j = 1, 2, ..., m$, present samples of other nature. Identify (in any way) this objects we come to empirical distribution

$$\pi_i(y) = \#\{z_{s_i} : z_{s_i} = y_i\}/m, \quad i = 1, 2, ..., q$$

and so on. It is evident π suggest higher tensor-rank characterization then y.

There is two interesting cases:

- It is possible to reduce π to a distribution on R_n similar to p_j. For example,

$$\pi_i = \sum P\{z_{s_l = y_l / x_{r_j}} = z_j\} p_j, \quad i = 1, 2, ..., q.$$

This leads us to conclusion, that for to construct total distribution function, we must found a transition (conditional) probabilities $p_{ij}(x)$ based on the distributions π_l and p_j.

- In the other hand, we become to derivative (non analytical, but covariant) of a distribution π_l with respect to distribution p_j of the type

$$(D_p\pi)_i = \pi_i(p) - \sum P_{ij}(x) p_j.$$

That is, on the sample space R_n we consider the tensor fields π_l and p_j. If $D_p\pi = 0$, the tensor fields π_l and p_j are called parallel. A set of such points have *geodesic* structure (absence of parametrization) and presents a strata for a given parallel tensor fields π_i and p_j. The strata and relevant collection of distributions in pairs form a *fiber*. Unification of all fibers transform the sample space into a fiber bundle. If $z_j(x) = z_j(y)$, under *parallel displacement* of tensors from x to y we mean the distribution

$$p_j(z) = \#\{x_{r_j} : x_{r_j} = z_j(y)\}/n.$$

The law of statistical inference may be formulated as follows:

- If sample space is one directed (one strata case), all empirical distributions are equivalent , probability model is uniquely defined and freely may be chosen total distribution among empirical distributions.

- In many strata case we do a parallel displacement and construct conditional probability law of π_i and p_j in role of total distribution (stratified sample space).

5. Remarks. Concerning so called kernel statistics, also, generalized histogram, it is remarkable, that the Parzen [3] and Nadaraya [4] type theorems are proved under the burdensome differential restrictions type supremum norm, integrability and so on. Sacrificing with probability (or, probability 1) convergence, which is incomprehensible for the quantum manifold, we are given the main theorem in frame of presented paper.

Theorem. *Let a sample space be a topological manifold. If for any coordinate neighborhood there exists local probability distribution - tensor pinned at any point of manifold, then there exist a global distribution on a manifold, like to conditional probability having covariant character.*

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