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THIRD ORDER DECOMPOSITION SCHEME FOR QUASI-LINEAR EVOLUTION EQUATION WITH VARIABLE OPERATOR

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Abstract. In the present work there is considered Caushy problem for abstract quasi-linear evolution equation with variable operator. For the considered problem the third order decomposition scheme is constructed and the convergence theorem for approximate solution is proved.

Keywords and phrases: Decomposition scheme, abstract evolution equation with variable operator, rational approximation.

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1. Introduction. Numerical computation of multi-dimensional problems using direct methods requires large computational time and resources. The most widespread and effective method for numerical calculations of multi-dimensional problems is decomposition (operator-splitting) method. Using the operator-splitting method, approximate solution of multi-dimensional problem can be deduced to solutions of one-dimensional problems. Namely, approximate solution of multi-dimensional problem can be constructed by solutions of one-dimensional problems, that significantly reduces time of numerical computation and makes the algorithm effective.

In the present work there is constructed the third order decomposition scheme for quasi-linear (with Lipshitz continuous operator) abstract evolution equation with variable operator. The third order operator splitting scheme for linear case is constructed in [1]-[3] and we generalize the same approach for quasi-linear equaion with variable operator. The convergence theorem for approximate solution is proved.

2. Statement of the problem. Let us consider the following problem:

$$\begin{aligned} \frac{du(t)}{dt} + b(t)\,Au(t) + M(u(t)) &= f(t), \quad t>0, \\ u(0) &= \varphi, \end{aligned} \tag{1}$$

where A is a self-adjoint, positive definite (generally unbounded) operator with the definition domain $D(A)$, which is everywhere dense in the space H, $b(t) \geq b_0 > 0$ is a given positive continuous scalar function; φ is a given vector from the definition domain $D(A)$, $f(t)$ is a continiuously differentiable function, nonlinear operator $M(\cdot)$ satisfies to Lipschitz condition.

Let A can be represented as a sum of m self-adjoint positive definite operators $A = A_1 + ... + A_m$.

Let us consider the following problem:

$$\frac{du(t)}{dt} + b(t)\,Au(t) = \widetilde{f}(t, u(t)), \quad t>0, \quad u(0) = \varphi, \tag{2}$$

where $\widetilde{f}(t,u(t)) = f(t) - M(u(t))$. If the above-mentioned conditions are satisfied for the problem (1), then the problem (2) has the unique solution and it is given by the following formula (see [4],[5]).

$$u(t) = U(t,0,A)\,u(0) + \int_0^t U(t,s,A)\,\widetilde{f}(s,u(s))\,ds, \tag{3}$$

where $U(t,s,A)$ is a solving operator of the problem (2), for which the following formula is valid (see [4],[5]):

$$U(t,s,A) = I - \int_s^t A(s_1)\,U(t,s_1,A)\,ds_1, \tag{4}$$

where $A(s_1) = b(s_1)\,A$.

using the formula (4) recursively we can obtain the following expansion for $U(t,s,A)$:

$$\begin{aligned} U(t,s,A) &= I - \int_s^t A(s_1)\,ds_1 + \int_s^t A(s_1)\int_s^t A(s_2)\,ds_2ds_1 \\ &+ ... + (-1)^{k-1}\int_s^t A(s_1)\int_s^{s_1} A(s_2)\,...\int_s^{s_{k-2}} A(s_{k-1})\,ds_{k-1}...ds_2ds_1 \\ &+ (-1)^k R_k(t,s,A), \end{aligned} \tag{5}$$

where for the residual $R_k(t,s,A)$ the following representation is valid:

$$R_k(t,s,A) = \int_s^t A(s_1)\int_s^{s_1} A(s_2)\,...\int_s^{s_{k-1}} A(s_k)\,U(t,s_k,A)\,ds_{k-1}...ds_2ds_1 \tag{6}$$

For the residual term (6) the following estimate is valid:

$$\|R_k(t,s,A)\,\varphi\| \leq c\,(t-s)^k\,\|\varphi\|_{A^k}, \tag{7}$$

where $\|\varphi\|_{A^k}$ can be defined recursively:

$$\begin{aligned} \|\varphi\|_{A^k} &= \|A_1\varphi\|_{A^{k-1}} + \|A_2\varphi\|_{A^{k-1}} + ... + \|A_m\varphi\|_{A^{k-1}}, \quad k = 1,2,..., \\ \|\varphi\|_{A^0} &= \|\varphi\|. \end{aligned}$$

Let us introduce the following mesh with respect to time variable

$$\overline{\omega}_\tau = \{t_k = k\tau, \quad k = 0,1,..., \quad \tau > 0\}.$$

To construct the decomposition scheme, we rewrite the formula (3) for the interval $[t_{k-1}, t_{k+2}]$. We obtain the formula:

$$u(t_{k+2}) = U(t_{k+2}, t_{k-1}, A)\,u(t_{k-1}) + \int_{t_{k-1}}^{t_{k+2}} U(t_{k+2}, s, A)\,\widetilde{f}(s,u(s))\,ds, \tag{8}$$

3. Decomposition scheme. Let us construct the third order decomposition scheme for Caushy problem of quasi-linear evolution equation with variable operator (2). For sake of simplicity we consider the case of $m = 2$ addends and then generalize it for any finite number of addends. We replace the solving operator $U(t_{k+2}, t_{k-1}, A)$ by locally fourth order splitting, and for integral part we use the locally fourth order quadrature formula. We obtain the third order accuracy decomposition scheme:

$$\begin{aligned} u_{k+2} &= V(t_{k+2}, t_{k-1})\, u_{k-1} \\ &+ \frac{3\tau}{4}\left(3V(t_{k+2}, t_{k+1})\, \widetilde{f}(t_{k+1}, u_{k+1}) + V(t_{k+2}, t_{k-1})\, \widetilde{f}(t_{k+1}, u_{k-1})\right), \end{aligned} \tag{9}$$

where the operator $V(t,s)$ is defined by the following formula:

$$\begin{aligned} V(t,s) &= \frac{1}{2}\left(W(t,s;\alpha A_1)\, W(t,s;\alpha A_2)\, W(t,s;\overline{\alpha} A_1)\right. \\ &\quad \left. + W(t,s;\alpha A_2)\, W(t,s;\alpha A_1)\, W(t,s;\overline{\alpha} A_2)\right), \end{aligned} \tag{10}$$

where $W(t,s;A)$ is a locally fourth order rational approximation of the operator $U(t,s;A)$ and is defined by the following formula:

$$W(t,s,A) = (I + \lambda_{0,t,s}(t-s)A)(I + \lambda_{t,s}(t-s)A)^{-1}\left(I + \overline{\lambda}_{t,s}(t-s)A\right)^{-1}. \tag{11}$$

Here the numerical parameters $\lambda_{0,t,s}$ and $\lambda_{t,s}$ are defined by the following formulas ($\lambda_{t,s}$ is a complex conjugate of $\lambda_{t,s}$):

$$\begin{aligned} \lambda_{0,t,s} &= \frac{6\gamma_{1,t,s}^3 - 6\gamma_{1,t,s}\gamma_{2,t,s} + \gamma_{3,t,s}}{6\gamma_{1,t,s}^2 - 3\gamma_{2,t,s}}, \quad \lambda_{t,s} = \frac{1}{2}\left(d_{t,s} + i\sqrt{4e_{t,s} - d_{t,s}^2}\right), \\ d_{t,s} &= \frac{3\gamma_{1,t,s}\gamma_{2,t,s} - \gamma_{3,t,s}}{6\gamma_{1,t,s}^2 - 3\gamma_{2,t,s}}, \quad e_{t,s} = \frac{3\gamma_{2,t,s}^3 - 3\gamma_{1,t,s}\gamma_{3,t,s}}{2\left(6\gamma_{1,t,s}^2 - 3\gamma_{2,t,s}\right)}, \\ \gamma_{1,t,s} &= \frac{3b(t) + b\left(s + \frac{t-s}{3}\right)}{4}, \quad \gamma_{2,t,s} = b^2\left(\frac{t+s}{2}\right), \quad \gamma_{3,t,s} = b^3\left(\frac{t+s}{2}\right). \end{aligned} \tag{12}$$

For numerical realization of the decomposition scheme (9) we need three starting vectors: u_0, u_1 and u_2. $u_0 = \varphi$ is defined from the initial condition. Computation of u_1 and u_2 by the third order accuracy is carried out by the equation:

$$u_i = V(t_i, t_{i-1})\, u_{i-1} + \frac{\tau}{2}\left(3V(t_i, t_{i-1})\, \widetilde{f}(t_{i-1}, u_{i-1}) + \widetilde{f}(t_i, u_i)\right), \quad i = 1, 2. \tag{13}$$

The right-hand side of this equation contains the unknown vector u_i, therefore for numerical realization of the scheme (13) it is necessary to use the following iteration:

$$u_i^{(l+1)} = F_i + \frac{\tau}{2} M\left(u_i^{(l)}\right), \quad i = 1, 2. \tag{14}$$

where l is an iteration index, and F_i is defined by the following formula:

$$F_i = V(t_i, t_{i-1})\, u_{i-1} + \frac{\tau}{2}\left(3V(t_i, t_{i-1})\, \widetilde{f}(t_{i-1}, u_{i-1}) + f(t_i)\right). \tag{15}$$

The iteration (14) converges quite fast, as the unknown vector in the right-hand side has small parameter and the operator $M(\cdot)$ satisfies to Lipschitz condition. Taking into account these two facts, we can easily show that the iterative process (13) converges with the rate of geometric progression, where common ratio is equal to $\frac{\tau}{2}$, multiplied on Lipschitz constant. To even accelerate the iteration process, the initial iteration should be taken equal to the solution on the previous time layer: $u_i^{(0)} = u_{i-1}$.

4. Convergence theorem. Let us state the theorem on the convergence of decomposition scheme (9).

Theorem 1. *Let the following conditions are satisfied:*

a) A, A_1 *and* A_2 *are self-adjoint, positive definite operators in the Hilbert space* H*;*

b) $b(t)$ *is a positive, scalar, three times continuously differentiable function* $b(t) \geq b_0 > 0$, $b(t) \in C^3[0,\infty)$*;*

c) The operator $M(\cdot)$ *satisfies to Lipschitz condition;*

d) $u(t) \in D(A^4)$ *for every* $t \geq 0$*;*

e) $\widetilde{f}(t, u(t)) \in C^3[0,\infty; H)$*;* $\widetilde{f}(t, u(t)) \in D(A^3)$, $\widetilde{f}'(t, u(t)) \in D(A^2)$ *and* $\widetilde{f}''(t, u(t)) \in D(A)$ *for every* $t \geq 0$*;*

Then for the error of the approximate solution the estimate holds:

$$\|u(t_k) - u_k\| = O\left(\tau^3\right).$$

Proof of the theorem is based on the following auxiliary lemmas:

Lemma 1. *If the (a) and (b) conditions of Theorem 1 are fulfilled, then for the operator the following expansion is valid:*

$$W(t, s; A) = \sum_{j=0}^{k-1} (-1)^j \frac{t^j}{j!} \gamma_{j,t,s} A^j + R_{W,k}(t, s; A), \quad k = 1, 2, 3, 4.$$

where $\gamma_{0,t,s} = 1$, *and* $\gamma_{j,t,s}$, $j = 1, 2, 3$ *are defined from formulas (12). Besides, for* $R_W(t, s; A)$ *residual term the estimate holds:*

$$\|R_{W,k}(t, s; A)\varphi\| \leq c e^{c(t-s)} (t-s)^4 \left\|A^k \varphi\right\|, \quad \varphi \in D\left(A^k\right).$$

Lemma 2. *If the conditions of Theorem 1 is fulfilled then the operator splitting* $V(t, s)$ *approximates the operator* $U(t, s, A)$ *by the locally fourth order accuracy, namely, the following estimate holds true:*

$$\|(U(t, s; A) - V(t, s))\varphi\| = O(t-s)^4, \quad \varphi \in D\left(A^4\right).$$

5. Generalization of decomposition scheme for case of any finite number of addends

Let us note that generalization of decomposition scheme (9) can be easily done for case of any finite number of addends:

$$A = A_1 + ... + A_m, \quad m > 2.$$

In this case, in the decomposition scheme (9) the operator splitting is defined by the following formula, instead of formula (10):

$$
\begin{aligned}
V(t,s) &= \frac{1}{2}\left(W(t,s;\alpha A_1)\,W(t,s;\alpha A_2)\ldots W(t,s;\alpha A_{m-1})\,W(t,s;A_m)\right. \\
&\times W(t,s;\overline{\alpha}A_{m-1})\,W(t,s;\overline{\alpha}A_{m-2})\ldots W(t,s;\overline{\alpha}A_1) \\
&+W(t,s;\alpha A_m)\,W(t,s;\alpha A_{m-1})\ldots W(t,s;\alpha A_2)\,W(t,s;A_1) \\
&\times W(t,s;\overline{\alpha}A_1)\,W(t,s;\overline{\alpha}A_2)\ldots W(t,s;\overline{\alpha}A_m)\,.
\end{aligned}
$$

where the rational approximation is defined by the formula (11) .

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R E F E R E N C E S

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