Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 27, 2013

ON THE OPTIMAL STOPPING OF PARTIALLY OBSERVABLE PROCESSES

Babilua P., Dochviri B., Purtukhia O., Sokhadze G.

Abstract. The Kalman-Buces continuous model of partially observable stochastic processes is considered. The problem of optimal stopping of a stochastic process with incomplete data is reduced to the problem of optimal stopping with complete data. The convergence of payoffs is proved when $\epsilon_1 \to 0$, $\epsilon_2 \to 0$, where ϵ_1 and ϵ_2 are small perturbation parameters of the nonobservable and observable processes respectively.

Keywords and phrases: Partially observable process, gain function, payoff, stopping time, optimal stopping.

AMS subject classification: 60J05, 91B28, 91B70.

1. Introduction. We consider a partially observable stochastic process (θ_t, ξ_t), $0 \le t \le T$, of Kalman-Buces model

$$d\theta_t = [a_0(t) + a_1(t)\theta_t]dt + \epsilon_1 dw_1(t), \tag{1}$$

$$d\xi_t = d\theta_t dt + \epsilon_2 dw_2(t), \tag{2}$$

where $\epsilon_1 > 0$, $\epsilon_2 > 0$ are constants, the coefficients $a_i(t)$, $i = 0, 1$, nonrandom measurable functions and $w_1(t)$, $w_2(t)$ are independent Wiener processes. It is assumed that in model (1), (2) θ_t is the nonobservable process and ξ_t is the observable process [1].

Consider a linear gain function of such from

$$g(x,t) - f_1(t) + f_2(t)x, \tag{3}$$

where $f_i(t)$, $i = 1, 2$, is nonrandom measurable function, $x \in R$, and introduce the payoffs

$$S_T^0 = sup_{\tau \in \Re_T^\theta} Eg(\tau, \theta_\tau), \quad S_T^{\epsilon_1,\epsilon_2} = sup_{\tau \in \Re_T^\xi} Eg(\tau, \theta_\tau), \tag{4}$$

where as usual we denote a class of all stopping times for a random process $X = (X_t, \Im_t^X)$ relative to a family of σ-algebras $F^X = (\Im_t^X)$ with $\Im_t^X = \sigma\{X_s, 0 \le s \le t\}$ as $\Re_T^X$ [1],[2].

The payoff S_T^0 corresponds to an optimal stopping problem with complete data for the process θ_t, while the payoff $S_T^{\epsilon_1,\epsilon_2}$ corresponds to the process θ_t with incomplete data. The first problem (reduction problem) consists in reducing the optimal stopping problem with incomplete data of the process θ_t to the optimal stopping problem of some completely observable process. The second problem (convergence of payoffs problem) is a proof the convergence $S_T^{\epsilon_1,\epsilon_2} \to S_T^0$ as $\epsilon_1 \to 0$, $\epsilon_2 \to 0$ [3], [4], [5].

Consider the example which show that from the smallness of coefficients ϵ_1 and ϵ_2 not necessarily by follows the closeness of the payoffs. We suppose that $\theta_t = \epsilon_1 w_1(t)$, $g(t,x) = g(x) = a$, when $x = x_0$ and $g(t,x) = 0$, when $x \neq x_0$, $x_0 \neq 0$. Then it is possible to show that $S_T^{\epsilon_1,\epsilon_2} \to 0 \neq S_T^0 = a$, when $\epsilon_1 \to 0$, $\epsilon_2 \to 0$.

In this paper the problems of reduction and convergence of payoff are investigated for model (1),(2).

2. The reduction problem. Let us introduce the following notations

$$m_t = E(\theta_t|\Im_t^\xi), \quad \gamma_t = E(\theta_t - m_t)^2. \tag{5}$$

Theorem 1. *The payoff $S_T^{\epsilon_1,\epsilon_2}$ can be presented in the following form*

$$S_T^{\epsilon_1,\epsilon_2} = sup_{\tau\in\Re_T^\xi} Eg(\tau, m_\tau). \tag{6}$$

Proof. Note that for arbitrary $\tau \in \Re_T^\xi$ and $A \in \Im$ we have $A \cap \{\tau \le t\} \in \Im_T^\xi$ for all $t \le T$. Because we have

$$S_T^{\epsilon_1,\epsilon_2} = sup_{\tau\in\Re_T^\xi} E\{f_1(\tau) + f_2(\tau)\theta_\tau\} = S_T^{\epsilon_1,\epsilon_2} = sup_{\tau\in\Re_T^\xi} E\{E[f_1(\tau) + f_2(\tau)\theta_\tau]|\Im_\tau^\xi\}$$

$$= S_T^{\epsilon_1,\epsilon_2} = sup_{\tau\in\Re_T^\xi} E\{f_1(\tau) + f_2(\tau)E(\theta_\tau|\Im_\tau^\xi)\}.$$

Next we can write

$$I_{\{\tau=t\}}E(\theta_\tau|\Im_\tau^\xi) = E(I_{\{\tau=t\}}\theta_\tau|\Im_\tau^\xi) = E(I_{\{\tau=t\}}\theta_t|\Im_\tau^\xi) = I_{\{\tau=t\}}E(\theta_t|\Im_\tau^\xi),$$

where I_A is the indicator of set A. According to Lemma 1.9[1], on the set $\{\tau = t\}$, we have $E(\theta_t|\Im_\tau^\xi) = E(\theta_t|\Im_t^\xi)$, i.e.

$$I_{\{\tau=t\}}E(\theta_\tau|\Im_\tau^\xi) = I_{\{\tau=t\}}E(\theta_t|\Im_t^\xi).$$

Thus we get the proof of (6).

Theorem 2. *The payoff $S_T^{\epsilon_1,\epsilon_2}$ can be presented in the following form*

$$S_T^{\epsilon_1,\epsilon_2} = sup_{\tau\in\Re_T^\theta} Eg(\tau, \widetilde{\theta}_\tau), \tag{7}$$

where the stochastic process $\widetilde{\theta}_t$ is defined by the following stochastic differential equation

$$d\widetilde{\theta}_t = [a_0(t) + a_1(t)\widetilde{\theta}_t]dt + a_1(t)\gamma_t dw_1(t). \tag{8}$$

Proof. According to Theorem 10.3 [1] and Theorem 7.12 [1] we have

$$dm_t = [a_0(t) + a_1(t)m_t]dt + a_1(t)\gamma_t(d\xi_t^\epsilon - [a_0(t) + a_1(t)m_t]dt),$$

$$d\xi_t = [a_0(t) + a_1(t)m_t]dt + \sqrt{\epsilon_1^2 + \epsilon_2^2}d\overline{w}(t), \tag{9}$$

$$dm_t = [a_0(t) + a_1(t)m_t]dt + \frac{a_1(t)\gamma_t}{\sqrt{\epsilon_1^2 + \epsilon_2^2}}d\overline{w}(t), \tag{10}$$

where $\overline{w}(t)$ is so called innovation Wiener process, which has such property that the σ-algebra $\Im_t^\xi$ and $\Im_t^{\overline{w}}$ coincide. From (8) and (10) we have

$$d\theta_t = \Phi_t[\int_0^t \Phi_s^{-1}a_0(s)ds + \int_0^t \Phi_s^{-1}\frac{a_1(s)\gamma_s}{\sqrt{\epsilon_1^2 + \epsilon_2^2}}dw_1(s)], \tag{11}$$

$$dm_t = \Phi_t[\int_0^t \Phi_s^{-1} a_0(s)ds + \int_0^t \Phi_s^{-1} \frac{a_1(s)\gamma_s}{\sqrt{\epsilon_1^2+\epsilon_2^2}} d\overline{w}(s)], \tag{12}$$

where the deterministic function Φ_t is defined by the following relation

$$\Phi_t = exp\{\int_0^t a_1(s)ds\}. \tag{13}$$

From (11), (12) we can write

$$sup_{\tau\in\Re^{\widetilde{\theta}}} Eg(\tau,\widetilde{\theta}_\tau) = sup_{\tau\in\Re^{\xi}} Eg(\tau,m_\tau), \tag{14}$$

where $\Re^{\widetilde{\theta}} = \Re^{\theta}$. Thus

$$sup_{\tau\in\Re^{\widetilde{\theta}}} Eg(\tau,\widetilde{\theta}_\tau) = sup_{\tau\in\Re^{\theta}} Eg(\tau,\widetilde{\theta}_\tau).$$

According to Theorem 1 $sup_{\tau\in\Re^{\xi}} Eg(\tau,m_\tau) = S_T^{\epsilon_1,\epsilon_2}$ and we get (7).

3. Convergence of payoffs. In proving the payoffs convergence rate, an estimation of the conditional variance γ_t by means of small parameters ϵ_1, ϵ_2 plays an essential role. We recall that for γ_t we have the ordinary differential equation

$$\gamma_t^{'} = 2a_1(t)\gamma_t - \frac{a_1^2(t)\gamma_t^2}{\epsilon_1^2+\epsilon_2^2} + \epsilon_1^2, \quad \gamma_0 = 0. \tag{15}$$

Let $\rho(t)$ denote a continuous increasing majorant of the function

$$\varphi(t) = \frac{\epsilon_1}{a_1(t)} \Phi_t^{-2},$$

where the function Φ_t is defined by (13).

Theorem 3. *Let $\rho(t) \geq \varphi(t)$. Then the following estimate holds for all $0 \leq t \leq T$:*

$$\gamma_t \leq \sqrt{\epsilon_1^2+\epsilon_2^2}\Phi_t^2 \rho(t). \tag{16}$$

Proof. We introduce a function u_t by using the following transformation

$$\gamma_t = \sqrt{\epsilon_1^2+\epsilon_2^2}\Phi_t^2 u_t, \quad u_0 = 0. \tag{17}$$

It is not difficult to see that the function u_t satisfies the ordinary differential equation

$$u_t^{'} = \frac{a_1^2(t)\Phi_t^2}{\sqrt{\epsilon_1^2+\epsilon_2^2}}[\frac{\epsilon_1^2\Phi_t^{-4}}{a_1^2(t)} - u_t^2], \quad u_0 = 0. \tag{18}$$

Let us show that $u_t \leq \rho(t)$, $0 \leq t \leq T$. Assume the opposite. Then there exist points t_0 and t_1 with $t_0 < t_1$ such that $u_{t_0} = \rho(t_0)$ and $u_t > \rho(t)$ for $t_0 < t \leq t_1$. For $t \in [t_0, t_1]$ we have

$$u_t^{'} \leq \frac{a_1^2(t)\Phi_t^2}{\sqrt{\epsilon_1^2+\epsilon_2^2}}[\rho^2(t) - u_t^2] < 0.$$

Therefore $u_t < u_{t_0} = \rho(t_0) \leq \rho(t)$ and we have obtained $u_t < \rho(t)$, which contradicts our assumption. Thus $u_t \leq \rho(t)$, $0 \leq t \leq T$, and we obtain the estimate (16).

We introduce the notations [5]:

$$h(t) = \epsilon_1^2 \int_0^t \Phi_s^{-2} ds, \quad \widetilde{h}(t) = \int_0^t \Phi_s^{-2} \frac{a_1^2(s)\gamma_s^2}{\sqrt{\epsilon_1^2 + \epsilon_2^2}} ds, \tag{19}$$

$$l = exp\{2 \int_0^T a_1(s)ds\}\rho(T), \tag{20}$$

$$Lg(t,x) = f_1^{'}(t) + f_2^{'}(t)x + f_2(t)[a_0(t) + a_1(t)x]. \tag{21}$$

Theorem 4. *Let the following condition hold:*

$$E(sup_{t\leq T} g(t,\theta_t)) < \infty. \tag{22}$$

Then the estimate is true:

$$0 \leq S_T^0 - S_T^{\epsilon_1,\epsilon_2} \leq (\epsilon_1 + \epsilon_2) l sup_{t\leq T} E(Lg(t,\theta_t)). \tag{23}$$

Proof. First we show that $S_T^0 \geq S_T^{\epsilon_1,\epsilon_2}$. from Theorem 3[5] and the identity of the σ-algebras $\Im_t^{\overline{w}}$ and $\Im_t^{\xi}$ it follows that

$$S_T^{\epsilon_1,\epsilon_2} = sup_{\tau\in\Re^{\overline{w}}} Eg(\tau, m_\tau + \eta\sqrt{\gamma_\tau}), \tag{24}$$

where η is standard normal random variable. The process m_t, $0 \leq t \leq T$, is Markovian with respect to the family $F^{\overline{w}} = (\Im_t^{\overline{w}})$ and in that case as is well known, the class of stopping times $\Im_T^m$ is sufficient [2], i.e. we have

$$S_T^{\epsilon_1,\epsilon_2} = sup_{\tau\in\Re_T^m} Eg(\tau, m_\tau + \eta\sqrt{\gamma_\tau}).$$

Let us now introduce an auxiliary payoff for stopping times $\tau \in \Re_T^\theta$:

$$\widetilde{S}_T^{\epsilon_1,\epsilon_2} = sup_{\tau\leq T_{\epsilon_1,\epsilon_2}} Eg(\tau,\theta_\tau), \tag{25}$$

where $T_{\epsilon_1,\epsilon_2}$ be denoted by the relation $\widetilde{h}(T) = h(T_{\epsilon_1,\epsilon_2})$. It is easy to see that for $\tau \in \Im_t^\theta$:

$$0 \leq S_T^0 - \widetilde{S}_T^{\epsilon_1,\epsilon_2} \leq sup_{\tau\leq T} E[g(\tau,\theta_\tau) - g(\tau\wedge T_{\epsilon_1,\epsilon_2}, \theta_{\tau\wedge T_{\epsilon_1,\epsilon_2}})],$$

where $s \wedge t := min(s,t)$.

Further, by Ito's formula we can write

$$E[g(\tau,\theta_\tau) - g(\tau\wedge T_{\epsilon_1,\epsilon_2}, \theta_{\tau\wedge T_{\epsilon_1,\epsilon_2}})] = E\int_{\tau\wedge T_{\epsilon_1,\epsilon_2}}^{\tau} Lg(t,\theta_t)dt \leq \int_{T_{\epsilon_1,\epsilon_2}}^{T} E[Lg(t,\theta_t)]dt$$

$$\leq (T - T_{\epsilon_1,\epsilon_2}) sup_{t\leq T} E[g(t,\theta_t)] \leq (\epsilon_1 + \epsilon_2) l sup_{t\leq T} E[g(t,\theta_t)].$$

Therefore we have

$$S_T^0 - \widetilde{S}_T^{\epsilon_1,\epsilon_2} \leq (\epsilon_1 + \epsilon_2) l sup_{t\leq T} E[g(t,\theta_t)]. \tag{26}$$

From (26), by Theorem 4 [5], we obtain the estimate (23).

Example. Consider the following model

$$d\theta_t = b(t)dw_1(t), \quad d\xi_t = A(t)w_1(t)dt + \epsilon dw_2(t).$$

We have

$$\gamma_t^{'} = b^2(t) - \frac{A^2(t)}{\epsilon^2}\gamma_t^2, \quad \gamma_t = \epsilon\frac{b(t)}{A(t)}th\frac{A(t)b(t)}{\epsilon}t,$$

where thx is tangens hyperbolic function of x. Let $Lg(t,x) = f_1^{'}(t) + f_2^{'}(t)x$.
The estimate (23) we can rewrite by following form:

$$0 \leq S_T^0 - S_T^\epsilon \leq \epsilon\frac{b(T)}{A(T)}th\frac{A(T)b(T)}{\epsilon}Tsup_{t\leq T}E[Lg(t,\theta_t)].$$

Acknowledgement. The work is supported by Shota Rustaveli National Science Foundation Grant No FR/308/5-104/12, FR/69/5-104/12.

REFERENCES

1. Liptser R.Sh., Shiryaev A.N. Statistics of Random Processes. *Vol. 1-2, Springer-Verlag, Berlin, New York*, 1977, 1978.

2. Shiryaev A.N. Optimal Stopping Rules. *Springer-Verlag, New York*, 1978.

3. Dochviri B. On optimal stopping with incomplete data. Probability Theory and Mathematical Statistics (Kyoto, 1986) *Lecture Notes in Math. 1299, Springer-Verlag, Berlin*, 1988, 64-68.

4. Babilua P., Bokuchava I., Dochviri B., Shashiashvili M. Convergence of costs in an optimal stopping problem for a partially observable model. *Appl. Math. Inform. Mech.*, **11**, (2006), 6-11.

5. Babilua P., Bokuchava I., Dochviri B. The optimal stopping problem for the Kalman-Bucy scheme. *Theory Probab. Appl.*, **55**, 1 (2010), 133-142.

Received 25.05.2013; accepted 27.11.2013.

Authors' addresses:

P. Babilua, B. Dochviri, G. Sokhadze
Iv. Javakhishvili Tbilisi State University
Faculty of Exact an Natural Sciences
Department of Mathematics
2, University Str., Tbilisi 0186
Georgia
E-mail: p_ babilua@yahoo.com
grigol.sokhadze@tsu.ge

O. Purtukhia
Iv. Javakhishvili Tbilisi State University
Faculty of Exact an Natural Sciences
Department of Mathematics
2, University St., Tbilisi 0186
Georgia

A. Razmadze Mathematical Institute
Georgian National Academy of Science
1, M. Aleksidze St., Tbilisi 0193
Georgia
E-mail: omar.purtukhi@tsu.ge