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NUMERICAL CALCULATION OF A TWO-DIMENSIONAL STATIC KIRCHHOFF EQUATION

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Abstract. The Chipot method of solution of a nonlinear static integro-differential problem is considered for a two-dimensional body. The result of a computer experiment of this method is presented.

Keywords and phrases: Static Kirchhoff equation, Chipot method, MATLAB.

AMS subject classification: 35Q74, 65N, 65H05.

Formulation of the problem. Let us consider the following boundary value problem

$$\varphi\left(\int_\Omega \left(w_x^2 + w_y^2\right)\, dx\, dy\right)(w_{xx} + w_{yy}) = f(x,y), \tag{1}$$

$$(x,y) \in \Omega,$$

$$w(x,y)\big|_{\partial\Omega} = 0, \tag{2}$$

where $\Omega = \{(x,y) \mid 0 < x < \pi,\ 0 < y < \pi\}$, $\partial\Omega$ is the boundary of the domain Ω; $\varphi = \varphi(z)$, $f = f(x,y)$ are given functions and $w = w(x,y)$ is the sought function. It is assumed that $\varphi(z)$, $0 \le z < \infty$, is a continuously differential function that satisfies the condition

$$\varphi(z) > \alpha > 0, \quad 0 \le z < \infty. \tag{3}$$

Equation (1) describes the static state of a two-dimensional body. When $\varphi(z)$ is a linear function

$$\varphi(z) = az + b, \tag{4}$$

where $a, b > 0$, equation (1) is obtained by truncation of the time argument t in a two-dimensional oscillation equation based on the Kirchhoff theory [1]. The introduction of the function $\varphi(z)$ makes it possible not to restrict the consideration to Hooke's law in the stress-strained equation [2].

Method of solution. To find $w(x,y)$ we will use M. Chipot's approach [3], [4]. The function $w(x,y)$ is written in the form

$$w(x,y) = \lambda v(x,y), \tag{5}$$

where λ and $v = v(x,y)$ are respectively the parameter and the function to be found. Substituting (5) into (1) we obtain

$$\lambda\varphi\left(\lambda^2 \int_\Omega (v_x^2 + v_y^2)\, dx\, dy\right)(v_{xx} + v_{yy}) = f(x,y). \tag{6}$$

Without loss of generality, equation (6) is replaced by the following system of equations

$$v_{xx} + v_{yy} = f(x,y),$$
$$\lambda\varphi\left(\lambda^2 \int_\Omega (v_x^2 + v_y^2)\,dx\,dy\right) = 1.$$

As seen from (2) and (5), the function $v(x,y)$ vanishes on the boundary. Therefore for this function we have the boundary value problem

$$v_{xx} + v_{yy} = f(x,y), \tag{7}$$
$$v\big|_{\partial\Omega} = 0, \tag{8}$$

whereas the parameter $\lambda > 0$ is defined as a solution of the equation

$$\lambda\varphi(s\lambda^2) = 1, \tag{9}$$

where

$$s = \int_\Omega (v_x^2 + v_y^2)\,dx\,dy > 0. \tag{10}$$

Equation (9) has a unique real solution. The question as to the accuracy of the considered method is studied in [5].

Computer experiment. To show the effectiveness of the method, we have solved the test example where

$$\varphi(z) = 1 + z^2,$$
$$f(x,y) = -2\left(1 + \left(\frac{\pi^8}{45}\right)^2\right)\left(x(\pi - x) + y(\pi - y)\right).$$

An exact solution of this problem has the form

$$w(x,y) = xy(\pi - x)(\pi - y). \tag{11}$$

Problem (7),(8) was solved using the MATLAB software. We have used method of finite difference schemes and sweep method. Then the parameter s was calculated by means of (10).

Using the obtained value s we formed equation (9) which in the considered case is an algebraic equation of fifth order

$$s^2\lambda^5 + \lambda - 1 = 0.$$

The unique root of this equation is obtained using the Roots command. The exact and obtained values are $\lambda = 2.249144e$–05 and $\widetilde{\lambda} = 2,249148e$–05 , respectively. The error is $3.598638e$–11.

Further, formula (5) is realized, which eventually gives an approximate value of the function $w(x,y)$ which we de denote by $\widetilde{w}(x,y)$.

Since in the general case the function $v(x,y)$, constant s and parameter λ can be calculated by the computer with only certain accuracy, we constructed the following table:

y	w	$\widetilde{w}$
0,314159265358979	2,19082786644575	2,19090834084857
0,628318530717959	3,89480509590355	3,89484443753616
0,942477796076938	5,11193168837341	5,11182518513252
1,25663706143592	5,84220764385533	5,84193349426986
1,57079632679490	6,08563296234930	6,08529271241343
1,88495559215388	5,84220764385533	5,84197430273477
2,19911485751286	5,11193168837341	5,11192658882468
2,51327412287183	3,89480509590355	3,89500856313457
2,82743338823081	2,19082786644575	2,19106741372825

Here $x = 1,539380400259$, the step and number of iterations are $h = 0.031$ and $M = 80$, respectively. The maximal error is $4.4128e$–04 . If we take $h = 0.006$ and $M = 2000$ then the maximal error is $9.7409e$–06.

For some methods of solution of one-dimensional static Kirchhoff equations with computer calculations see for example [6], [7].

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