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NUMERICAL SOLUTION OF THE BOUNDARY VALUE PROBLEM OF STATICS IN THE THEORY OF THERMOELASTICITY WITH MICROTEMPERATURES FOR CIRCULAR RING

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Abstract. In this paper the solution of the boundary value problems of the theory of thermoelasticity with microtemperatures for the circular ring are considered.The representation of regular solution for the equations of the theory of thermoelasticity with microtemperatures by harmonic and metaharmonic functions is obtained, that we use for explicitly solving basic boundary value problems (BVPs) for the circular ring. The obtained solutions are represented as absolutely and uniformly convergent series.

Keywords and phrases: Thermoelasticity, microtemperature, boundary value problem, numerical solutions.

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Basic equations. The basic system of equations of the theory of thermoelasticity with microtemperatures can be written in the form [1,2]:

$$\begin{aligned} &\mu\Delta u(x) + (\lambda+\mu) graddivu(x) = \beta gradu_3(x), \\ &k\Delta u_3(x) + k_1 divw(x) = 0, \\ &k_6\Delta w(x) + (k_4+k_5) graddivw(x) - k_3 gradu_3(x) - k_2 w(x) = 0, \end{aligned} \tag{1}$$

where $\lambda, \mu, \beta, k, k_1, k_2, k_3, k_4, k_5, k_6$ are constitutive coefficients [1]; $u(x)$ is the displacement of the point $x = (x_1, x_2)$; $u = (u_1, u_2)$; $w = (w_1, w_2)$ is the microtemperature vector; u_3 is temperature measured from the constant absolute temperature T_0; Δ is the Laplace operator.

Problem. Find a regular vector $U = (u_1, u_2, u_3, w_1, w_2)$, $(U \in C^1(\overline{D}) \cap C^2(D), \overline{D} = D \cup S_0 \cup S_1)$ satisfying in the ring D a system of equations (1) and on the circumferences S_0 and S_1 the boundary conditions:

$$u^i(z) = f^i(z), \quad u_3^i(z) = f_3^i(z), \quad w^i(z) = p^i(z), \quad i = 0, 1, \tag{2}$$

where $f = (f_1, f_2)$, $\quad p = (p_1, p_2)$, $\quad f_1, f_2, f_3$ are the given functions on S_0 and S_1.

The above-formulated problem of thermoelasticity with microtemperature can be considered as a union of two problems - $A \quad and \quad B$, where:

Problem A - find in a ring D the solution $u(x)$ of equation $(1)_1$, if on the circumferences S_0 and S_1 there are given the values of the vector $u(z)$;

Problem B - find in the ring D the solutions $u_3(x)$ and $w(x)$ of the system of equations $(1)_2$ and $(1)_3$, if on the circumferences S_0 and S_1 there are given the values of the function $u_3(z)$ and of the vector $w(z)$.

Solution of the Problem B. By virtue of system $[(1)_2,(1)_3]$ and conditions (2), we can write [3]:

$$\begin{aligned} u_3(x) &= \varphi_1(x)+\varphi_2(x),\\ w_n(x) &= a_1\partial_r\varphi_1(x)+a_2\partial_r\varphi_2(x)-a_3\frac{1}{r}\partial_\psi\varphi_3(x),\\ w_s(x) &= a_1\frac{1}{r}\partial_\psi\varphi_1(x)+a_2\frac{1}{r}\partial_\psi\varphi_2(x)+a_3\partial_r\varphi_3(x); \end{aligned} \tag{3}$$

$$u_3^i(z)=f_3^i(z),\quad w_n^i(z)=p_n^i(z),\quad w_s^i(z)=p_s^i(z),\quad i=0,1, \tag{4}$$

where $\triangle\varphi_1=0,\ \ (\triangle+s_1^2)\varphi_2=0,\ \ (\triangle+s_2^2)\varphi_3=0,\ \ s_1^2=-\dfrac{kk_2-k_1k_3}{kk_7},\quad s_2^2=-\dfrac{k_2}{k_6},$

$a_1=-\frac{k_3}{k_2},\ \ a_2=-\frac{k}{k_1},\ \ a_3=\frac{k_6}{k_7};\quad k_7=k_4+k_5+k_6;\quad k,k_2,k_6,k_7>0;$

$w_n=(w\cdot n),\quad w_s=(w\cdot s),\quad p_n=(p\cdot n),\quad p_s=(p\cdot s),\quad n=(n_1,n_2),$

$s=(-n_2,n_1);\quad x=(r,\psi),\quad r^2=x_1^2+x_2^2.$

The harmonic function φ_1 and metaharmonic functions φ_2 and φ_3 are represented in the form of series in the ring [4,5]:

$$\begin{aligned} \varphi_1(x) &= X_{10}\ln r+Y_{10}+\sum_{m=1}^{\infty}[r^m(X_{1m}\cdot\nu_m(\psi))+r^{-m}(X_{1m}\cdot\nu_m(\psi))],\\ \varphi_2(x) &= \sum_{m=0}^{\infty}[I_m(s_2r)(X_{2m}\cdot\nu_m(\psi))+K_m(s_2r)(Y_{2m}\cdot\nu_m(\psi))],\\ \varphi_3(x) &= \sum_{m=0}^{\infty}[I_m(s_3r)(X_{3m}\cdot\ s_m(\psi))+K_m(s_3r)(Y_{3m}\cdot\ s_m(\psi))], \end{aligned} \tag{5}$$

where $I_m(s_jr)$ and $K_m(s_jr)$ are Bessel's and modified Hankel's functions of an imaginary argument, respectively; X_{km} and Y_{km} are the unknown two-component constants vectors, $\nu_m(\psi)=(\cos m\psi,\sin m\psi)$, $s_m(\psi)=(-\sin m\psi,\cos m\psi)$, $j=2,3,k=1,2$.

We substitute (5) into (3) and then the obtained expression into (4). Passing to the limit, as $r\to R_0$ and $r\to R_1$ for the unknowns X_{mk} and Y_{mk} we obtain a system of algebraic equations:

$$\begin{aligned} &\tfrac{1}{R_i}X_{10}+a_2s_2I_0'(s_2R_i)X_{20}+a_2s_2K_0'(s_2R_i)Y_{20}=\tfrac{A_{10}^i}{2},\\ &a_3s_3I_0'(s_3R_i)X_{30}+a_3s_3K_0'(s_3R_i)Y_{30}=\tfrac{A_{20}^i}{2},\\ &X_{10}\ln R_i+Y_{10}+I_0(s_2R_i)X_{20}+K_0(s_2R_i)Y_{20}=\tfrac{A_{30}^i}{2},\\ &mR_i^{m-1}X_{1m}-mR_i^{-(m+1)}Y_{1m}+a_2s_2I_m'(s_2R_i)X_{2m}+a_2s_2K_m'(s_2R_i)Y_{2m},\\ &-a_3\tfrac{m}{R_i}I_m(s_3R_i)X_{3m}-a_3\tfrac{m}{R_i}K_m(s_3R_i)Y_{3m}=A_{1m}^i,\\ &a_1mR^{m-1}X_{1m}+a_1mR^{m-1}Y_{1m}+a_2\tfrac{m}{R_i}I_m(s_2R_i)X_{2m}+a_2\tfrac{m}{R_i}K_m(s_2R_i)Y_{2m},\\ &+a_3s_3I_m'(s_3R_i)X_{3m}+a_3s_3K_m'(s_3R_i)Y_{3m}=A_{2m}^i,\\ &R_i^mX_{1m}+R_i^{-m}Y_{1m}+I_m(s_2R_i)X_{2m}+K_m(s_2R_i)Y_{2m}=A_{3m}^i,\quad i=0,1, \end{aligned} \tag{6}$$

where A^i_{1m}, A^i_{2m} and A^i_{3m} are the Fourier coefficients of the functions $p_n(z)$, $p_s(z)$ and $f_3(z)$, respectively.

Solution of the Problem A. The solution of the first equation of system (1) with the boundary condition (2) is represented by the sum

$$u(x) = v_0(x) + v(x), \tag{7}$$

where v_0 is a particular solution of equation $(1)_1$

$$v_0(x) = \frac{\beta}{\lambda + 2\mu} grad[-\frac{1}{s_1^2}\varphi_2(x) + \varphi_0(x)]; \tag{8}$$

φ_0 is a biharmonic function: $\triangle\varphi_0 = \varphi_1$; $v(x) = (v_1(x), v_2(x))$ is the solution of the homogeneous equation $\mu\triangle v(x) + (\lambda+\mu) graddivv(x) = 0$ which can be found by means of formula [3]

$$v_1(x) = \frac{\partial}{\partial x_1}[\Phi_1(x)+\Phi_2(x)] - \frac{\partial}{\partial x_2}\Phi_3(x), \quad v_2(x) = \frac{\partial}{\partial x_2}[\Phi_1(x)+\Phi_2(x)] + \frac{\partial}{\partial x_1}\Phi_3(x), \tag{9}$$

where $\Delta\Phi_1(x) = 0$, $\Delta\Delta\Phi_2(x) = 0$, $\Delta\Delta\Phi_3(x) = 0$;

$$\begin{aligned}
\Phi_1(x) &= \sum_{m=1}^{\infty}[\left(\frac{r}{R_1}\right)^m (Z_{1m}\cdot \nu_m(\psi)) + \left(\frac{R}{r}\right)^m (Z_{2m}\cdot\nu_m(\psi))] + Z_{10}\ln r,\\
\Phi_2(x) &= \sum_{m=0}^{\infty}\left(\frac{r}{R_1}\right)^{m+2} (Z_{3m}\cdot\nu_m(\psi))\\
&+ \sum_{m=2}^{\infty}\left(\frac{R_0}{r}\right)^{m-2}(Z_{4m}\cdot\nu_m(\psi)) + r\ln r(Z_{41}\cdot\nu_1(\psi)) + \frac{1}{2}\left(\frac{r}{R_1}\right)^2 Z_{20},
\end{aligned} \tag{10}$$

$$\Phi_3(x) = -\frac{(\lambda+2\mu)}{\mu}\sum_{m=1}^{\infty}\left(\frac{r}{R_1}\right)^{m+2}(Z_{3m}\cdot s_m(\psi)) + \frac{\lambda+2\mu}{\mu}\sum_{m=2}^{\infty}\left(\frac{R_0}{r}\right)^{m-2}(Z_{4m}\cdot s_m(\psi))$$
$$+ \frac{(\lambda+2\mu)}{\mu} r\ln r(Z_{11}\cdot s_1(\psi)) + Z_{40}\ln r + \frac{1}{2}\left(\frac{r}{R_1}\right)^2 Z_{30},$$

where Z_{km} are the unknown two-component vectors, $k = 1, 2, 3, 4$.

Taking into account (7) and relying on condition $(2)_I$, we can write

$$v^i(z) = \Psi^i(z), \tag{11}$$

where $\Psi^i(z) = f^i(z) - v_0(z)$ is the known vector. Substituting (10) into (9), the obtained expressions into (11), we obtain the system of algebraic equations for every m:

$$\begin{aligned}
&t_1 m t^{m-1} Z_{1m} - m t_0 Z_{2m} - e_1(m) t^{m-1} Z_{3m} + a(m) Z_{4m} = \eta^0_m,\\
&m t^1 Z_{1m} - m t_0 m t^{m+1} Z_{2m} - t_1 e_1(m) Z_{3m} + b(m) Z_{4m} = \eta^1_m,\\
&t^{m-1} Z_{1m} - m t_0 Z_{2m} - t_1 q_2(m) t^{m+1} - c(m) Z_{4m} = \varsigma^0_m,\\
&Z_{1m} + t_0 m t^{m+1} - t_1 q_2(m) Z_{3m} - d(m) Z_{4m} = \varsigma^1_m, \quad m = 1, 2, ...,
\end{aligned} \tag{12}$$

where $p_1 = mu(\lambda + 3\mu), p_2 = mu(\lambda + 2\mu)$,

$$a(1) = p_1 \ln R_0 + 1, \quad a(m) = t_0 e_2(m), m = 2, 3, ...; e_1(m) = mu[(\lambda + \mu) - 2\mu],$$

$$b(1) = p_1 \ln R_1 + 1, \quad b(m) = t_0 e_2(m) t^{m-1}, m = 2, 3, ...; e_2(m) = mu[(\lambda + \mu) + 2\mu],$$

$c(1) = p_1 \ln R_0 + p_2, \quad c(m) = t_0 q_1(m), m = 2, 3, ...; q_1(m) = mu[(\lambda + \mu)(m - 2) - 2\mu],$

$d(1) = p_1 \ln R_0 + 1, \quad d(m) = t_0 q_1(m) t^{m-1}, m = 2, 3, ...; q_2(m) = mu[(\lambda + \mu)(m + 2) + 2\mu],$

$t = \dfrac{R_0}{R_1}, t_0 = \dfrac{1}{R_0}, t_1 = \dfrac{1}{R_1}, mu = \dfrac{1}{\mu};$ η_m^i and ς_m^i are the Fourier coefficients of the functions Ψ_n^i and Ψ_s^i, respectively. If $m = 1$, then $Z_{10} = \dfrac{\Delta_1}{\Delta}, \quad Z_{20} = \dfrac{\Delta_2}{\Delta},$
$Z_{30} = -\dfrac{\Delta_1'}{\Delta}, \quad Z_{40} = \dfrac{\Delta_2'}{\Delta},$ where $\Delta = \dfrac{R_1^2 - R_0^2}{R_0 R_1^3} \neq 0, \quad \Delta_1 = \dfrac{1}{2R_1}\left(\eta_0^0 - \dfrac{R_0}{R_1}\eta_0^1\right),$
$\Delta_2 = \dfrac{1}{2}\left(\dfrac{\eta_0^1}{R_0} - \dfrac{\eta_0^0}{R_1}\right), \quad \Delta_1' = -\dfrac{1}{2}\left(\dfrac{\varsigma 1_0}{R_0} - \dfrac{\varsigma_0^0}{R_1}\right), \quad \Delta_2' = -\dfrac{1}{2R_1}\left(\varsigma_0^0 - \dfrac{R_0}{R_1}\varsigma_0^1\right).$

Numerical solutions. For the numerical solution there is the program. $w(x)$ and $u_3(x)$ are calculated from (3), (5) and (7); $u_1(x)$ and $u_2(x)$ are calculated from (6), where $v_0(x)$ calculated from (8), (6) and (5), while $v(x)$ from (10) and (12).

Let us consider a particular case with the following conditions:

$$R_0 = 2; \quad R_1 = 4; \quad r = 3; \quad \psi = 45°; \quad \lambda = 7.28 \cdot 10^6; \quad \mu = 3.5 \cdot 10^6; \quad k_1 = 0,4;$$

$$k_2 = 0.3; \quad k_3 = 0,4; k_4 = 1,1; k_5 = 0,5; k_6 = 0,22; k_7 = k_4 + k_5 + k_6; \beta_1 = 0.3;$$

$c = 0; \quad d = 2\pi;$
$f_1^0(\theta) = \dfrac{R_0}{2}(\cos\theta - \dfrac{1}{4}) \cdot 10^{-4}; f_2^0(\theta) = R_0(\sin\theta + 2) \cdot 10^{-4}; f_1^1(\theta) = \dfrac{R_1}{2}(\cos\theta - \dfrac{1}{4}) \cdot 10^{-4};$

$f_2^1(\theta) = R_1(\sin\theta + 2) \cdot 10^{-4}; \quad p_1^i = R_i(\sin(\theta) - 1)10^{-6}; \quad p_2^i = R_i(\sin(\theta) + 2)10^{-6};$

$f_3^i = \dfrac{1}{3}R_i(\cos(\theta) + 2)10^{-1}; \quad 0 \leq \theta \leq 2\pi.$

We obtain that:

$$u_1 = 1.432 \cdot 10^{-4}; \quad u_2 = -1.11 \cdot 10^{-3}; \quad w_1 = 0.534; \quad w_2 = -2.472; \quad u_3 = 8.656.$$

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