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ROBUST MEAN-VARIANCE HEDGING IN THE SINGLE PERIOD MODEL

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Abstract. We give an explicit solution of robust mean-variance hedging problem in the continuous time model for some type of contingent claims.

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We consider a financial market with one riskless asset of price $S^0 = 1$ and one risky asset of price process S. We shall fix throughout a canonical, filtered measurable space $(\Omega, \mathcal{F}, F = (F_{t+})_{t \in T})$, and assume that Ω coincides with the space $C[0,T]$ of all continuous functions, $F_t = \sigma(w(s), 0 \leq s \leq t)$ and $\mathcal{F} = F_T$. We shall call admissible system a collection M consisting of the underlying filtered space $(\Omega, \mathcal{F}, F)$, of a probability measure P on it, and of a pair of processes (S, W) , with W F-Brownian motion. These processes have the dynamics

$$dS_t = b_t(S)dt + \sigma_t(S)dW_t, \tag{1}$$

for some progressively measurable functionals b and σ with value in convex compact subset $K \subset R \times R_+$.

As proved by Krylov [2] the distribution laws of such type processes constitute weak compact convex subset $\mathcal{P}_\mathcal{K}$ in the set of probability measures on $(\Omega, \mathcal{F})$.

An agent, starting from a capital x, invests an amount π_t at any time t in the risky asset. His wealth process, controlled by X_t, is given by

$$X_t = x + \int_0^T \pi_u dS_u = x + \int_0^T \pi_u(b_u du + \sigma_u dW_u),\ 0 \leq t \leq T. \tag{2}$$

We denote by Π the set of progressively measurable processes π on $(\Omega, \mathcal{F}, F)$, such that

$$E^P \int_0^T |\pi_t|^2 dt < \infty \ for\ all\ P \in \mathcal{P}_K \tag{3}$$

and by U_K the set of progressively measurable processes (b, σ) on $(\Omega, \mathcal{F}, F)$ valued in K. The robust mean-variance hedging problem (also called robust quadratic minimization problem) of a contingent claim $H(S)$ is formulated as

$$\min_{\pi \in \Pi} \max_{(b,\sigma) \in U_K} E|H(S) - x - \int_0^T \pi_t dS_t|^2, \tag{4}$$

which by results of Krylov [2] and by a saddle point existence theorem of Neumann [1] can be reformulated as

$$\min_{\pi\in\Pi}\max_{P\in\mathcal{P}_K} E^P|H(S)-x-\int_0^T \pi_t dS_t|^2$$
$$=\max_{P\in\mathcal{P}_K}\min_{\pi\in\Pi} E^P|H(S)-x-\int_0^T \pi_t dS_t|^2$$
$$=\max_{(b,\sigma)\in U_K}\min_{\pi\in\Pi} E|H(S)-x-\int_0^T \pi_t dS_t|^2.$$

The mean-variance hedging problem with given coefficients

$$v_{\theta,\sigma}(x)=\min_{\pi\in\Pi} E|H(S)-x-\int_0^T \pi_t\sigma_t(\theta_t dt+dW_t)|^2, \qquad (5)$$

has solution of the form (see [3])

$$v_{\theta,\sigma}(x)=\frac{(x-EH(S)\mathcal{E}_T(-\int\theta dW))^2}{E\mathcal{E}_T^2(-\int\theta dW)}$$
$$\pi_t^*=h_t-\theta_t/\sigma_t(E(H(S)|\mathcal{F}_t)-X_t^*),$$

where $\theta=\frac{b}{\sigma}$, $\mathcal{E}_t(-\int\theta dW))$ is Doleans-Dade exponential and h is the integrand of the stochastic integral representation of $E(H(S)|\mathcal{F}_t)$. Suppose that

$$K=\{(b,\sigma):\theta=b/\sigma\in[\underline{\theta},\overline{\theta}],\sigma\in[\underline{\sigma},\overline{\sigma}]\},$$

where $\underline{\theta}\le\overline{\theta},\ 0<\underline{\sigma}\le\overline{\sigma}$ are given numbers. Hence

$$\max_{(b,\sigma)\in U_K}\min_{\pi\in\Pi} E|H(S)-x-\int_0^T \pi_t dS_t|^2$$
$$=\max_{(b,\sigma)\in U_K}\frac{(x-EH(S)\mathcal{E}_T(-\int\theta dW))^2}{E\mathcal{E}_T^2(-\int\theta dW)}$$
$$=\frac{\max_\sigma(x-EH(S)\mathcal{E}_T(-\int\theta dW))^2}{\min_\theta E\mathcal{E}_T^2(-\int\theta dW)}$$
$$=\frac{\max_\sigma(x-EH(S)\mathcal{E}_T(-\int\theta dW))^2}{e^{\underline{\theta}^2T}}.$$

Denote by $\tilde{E}$ the expectation with respect to $\tilde{P}=\mathcal{E}_t(-\int\theta dW))P$. Then S is the solution of $dS_t=\sigma_t(S)d\tilde{W}_t$ with respect to $\tilde{P}$. Since

$$\max_{\underline{\sigma}\le\sigma_t\le\overline{\sigma}}(x-\tilde{E}H(S))^2$$
$$=\begin{cases}(x-\max_{\underline{\sigma}\le\sigma_t\le\overline{\sigma}}\tilde{E}H(S))^2\ if\ x<\frac{1}{2}(\max_{\underline{\sigma}\le\sigma_t\le\overline{\sigma}}\tilde{E}H(S)+\min_{\underline{\sigma}\le\sigma_t\le\overline{\sigma}}\tilde{E}H(S)),\\ (x-\min_{\underline{\sigma}\le\sigma_t\le\overline{\sigma}}\tilde{E}H(S))^2\ if\ x>\frac{1}{2}(\max_{\underline{\sigma}\le\sigma_t\le\overline{\sigma}}\tilde{E}H(S)+\min_{\underline{\sigma}\le\sigma_t\le\overline{\sigma}}\tilde{E}H(S)),\end{cases}$$

the minimax problem reduces to the solution of problems

$$\max_{\underline{\sigma}\le\sigma_t\le\overline{\sigma}} \tilde{E}H(S) \; and \; \min_{\underline{\sigma}\le\sigma_t\le\overline{\sigma}} \tilde{E}H(S).$$

If H is a terminal functional i.e. $H(S)=g(S_T)$ for continuous function $g(s)$ satisfying linear growth condition, $\overline{v}(t,s)=\max_{\underline{\sigma}\le\sigma_t\le\overline{\sigma}} \tilde{E}_{t,s}g(S_T)$ satisfies the Barenblatt equation

$$\overline{v}_t(t,s)+\frac{1}{2}\overline{\sigma}\,\overline{v}_{ss}(t,s)^+-\frac{1}{2}\underline{\sigma}\overline{v}_{ss}(t,s)^-=0,\; \overline{v}(T,s)=g(s)$$

and

$$\sigma^*(t,s)=\begin{cases}\overline{\sigma} & if \;\; \overline{v}_{ss}(t,s)>0,\\ \underline{\sigma} & if \;\; \overline{v}_{ss}(t,s)<0.\end{cases}$$

Similarly for the value function $\underline{v}(t,s)=\min_{\underline{\sigma}\le\sigma_t\le\overline{\sigma}} \tilde{E}_{t,s}g(S_T)$ we have the equation

$$\underline{v}_t(t,s)+\frac{1}{2}\underline{\sigma}\;\underline{v}_{ss}(t,s)^+-\frac{1}{2}\overline{\sigma}\underline{v}_{ss}(t,s)^-=0,\; \underline{v}(T,s)=g(s)$$

and

$$\sigma^*(t,s)=\begin{cases}\underline{\sigma} & if \;\; \underline{v}_{ss}(t,s)>0,\\ \overline{\sigma} & if \;\; \underline{v}_{ss}(t,s)<0.\end{cases}$$

Hence we have proved

Theorem 1. *The saddle point* $(\pi^*,\theta^*,\sigma^*)$ *of the minimax problem is defined by the equation*

$$\theta^*_t=\underline{\theta},$$

$$\sigma^*_t=\begin{cases}\overline{\sigma} & if \;\; \overline{v}_{ss}(t,S^*_t)>0,\; x<\frac{1}{2}(\underline{v}(0,S_0)+\overline{v}(0,S_0)),\\ \underline{\sigma} & if \;\; \overline{v}_{ss}(t,S^*_t)<0,\; x<\frac{1}{2}(\underline{v}(0,S_0)+\overline{v}(0,S_0)),\\ \underline{\sigma} & if \;\; \underline{v}_{ss}(t,S^*_t)>0,\; x>\frac{1}{2}(\underline{v}(0,S_0)+\overline{v}(0,S_0)),\\ \overline{\sigma} & if \;\; \underline{v}_{ss}(t,S^*_t)<0,\; x>\frac{1}{2}(\underline{v}(0,S_0)+\overline{v}(0,S_0)),\end{cases}$$

$$\pi^*_t=h_t-\theta^*_t/\sigma^*_t(\tilde{E}(H(S^*)|\mathcal{F}_t)-X^*_t).$$

Corollary 1. If g is a convex function then the optimal pair is

$$(\theta^*_t,\sigma^*_t)=\begin{cases}(\underline{\theta},\overline{\sigma}) & if \;\; x<\frac{1}{2}(Eg(\overline{\sigma}W_T)+g(\underline{\sigma}W_T)),\\ (\underline{\theta},\underline{\sigma}) & if \;\; x>\frac{1}{2}(Eg(\overline{\sigma}W_T)+g(\underline{\sigma}W_T)).\end{cases}$$

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