Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 26, 2012

ON PARABOLIC REGULARIZATION FOR ONE NONLINEAR PARTIAL DIFFERENTIAL SYSTEM

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Abstract. Parabolic regularization of one-dimensional analog of the system of nonlinear partial differential equations arising in the process of vein formation of young leaves is considered. Numerical resolution of the initial-boundary value problems for this system is done by the finite difference schemes. Graphical illustrations of the tests experiments are given.

Keywords and phrases: One-dimensional system of nonlinear partial differential equations, vein formation model, parabolic regularization, finite difference scheme.

AMS subject classification: 35Q80, 65N06, 65Y99.

In [1] the mathematical model arising in the process of vein formation of young leaves is stated. On the rectangle $[0,1]\times[0,T]$ let us consider the following initial-boundary value problem for one-dimensional analog of this system

$$\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\left(V\frac{\partial U}{\partial x}\right), \tag{1}$$

$$\frac{\partial V}{\partial t}=-V+G\left(V\frac{\partial U}{\partial x}\right), \tag{2}$$

$$U(0,t)=U(1,t)=0, \tag{3}$$

$$U(x,0)=U_0(x), \quad V(x,0)=V_0(x)\geq Const>0, \tag{4}$$

where G, U_0, V_0 are known sufficiently smooth functions, $g_0\leq G(\xi)\leq G_0$; T, g_0, G_0, δ_0 are given positive constants.

Let us consider the parabolic regularization of problem (1)-(4):

$$\frac{\partial U^\varepsilon}{\partial t}=\frac{\partial}{\partial x}\left(V^\varepsilon\frac{\partial U^\varepsilon}{\partial x}\right), \tag{5}$$

$$\frac{\partial V^\varepsilon}{\partial t}=-V^\varepsilon+G\left(V^\varepsilon\frac{\partial U^\varepsilon}{\partial x}\right)+\varepsilon\frac{\partial^2 U^\varepsilon}{\partial x^2}, \tag{6}$$

$$U^\varepsilon(0,t)=U^\varepsilon(1,t)=\frac{\partial V^\varepsilon(0,t)}{\partial x}=\frac{\partial V^\varepsilon(1,t)}{\partial x}=0, \tag{7}$$

$$U^\varepsilon(x,0)=U_0(x), \quad V^\varepsilon(x,0)=V_0(x)\geq Const, \tag{8}$$

where ε is the positive constant.

It is well known that parabolic regularizations are studied for many mathematical and practical problems (see, for example, [2]).

Let us note that (5),(6) type parabolic approximations for (1),(2) system are considered in [3].

In [4] some qualitative and structural properties of solutions of the boundary-value problems for system (1), (2) are established. The system (1),(2) and its two-dimensional analog that are given in [1] are considered in many scientific works as well (see, for example, [5]-[17] and references therein). The questions of investigation and numerical resolution of some kind of initial-boundary value problems for models of this type are studied in these works.

The purpose of the present note is to study convergence of solution U^{ε}, V^{ε} of problem (5)-(8) to the solution U, V of the problem (1)-(4) as $\varepsilon \to 0$. Establishing this convergence by the numerical experiments is also the purpose of this note.

Let us assume that problems (1)-(4) and (5)-(8) have regular solutions. The following statement takes place.

Theorem. *The solution U^{ε}, V^{ε} of the problem (5)-(8) converges to the solution U, V of the problem (1)-(4) as $\varepsilon \to 0$ in the norm of the space $L_2(0,1)$.*

Let us consider problems (1)-(4) and (5)-(8) with the same nonhomogeneous right hand sides in both equations of the systems. Introduce the uniform grids $\bar{\omega}_h = \{t_i = ih, i = 0, 1, ...M\}$ on $[0,1]$ and $\omega_\tau = \{t_j = j\tau, j = 0, 1, ...N\}$ on $[0,T]$ and using usual notations (see, for example, [18]) let us consider the following finite difference scheme

$$u_t^{\varepsilon} = (\hat{v}_{\varepsilon} \hat{u}_{\bar{x}}^{\varepsilon})_x + f, \tag{9}$$

$$v_t^{\varepsilon} = \sigma G\left(\hat{v}^{\varepsilon} \hat{u}_{\bar{x}}^{\varepsilon}\right) + (1-\sigma) G\left(v^{\varepsilon} u_{\bar{x}}^{\varepsilon}\right) + \varepsilon \hat{v}_{\bar{x}x}^{\varepsilon} + g, \tag{10}$$

$$u_0^{\varepsilon,j} = u_M^{\varepsilon,j} = v_{x0}^{\varepsilon,j} = v_{\bar{x}M}^{\varepsilon,j} = 0 \quad j = 0, 1, .., N, \tag{11}$$

$$u_i^{\varepsilon,0} = U_{0,i} \quad v_i^{\varepsilon,0} = V_{0,i} \quad i = 0, 1, .., M. \tag{12}$$

Here f, g are given functions and $0 \le \sigma \le 1$ is a given constant. (9),(10) kind scheme for (5)-(8) type initial-boundary value problem with first type boundary conditions on the function V^{ε} are constructed and investigated in [17].

Note that, in the case $\sigma = 0$, for solving the finite difference scheme (9)-(12) at first we solve system (10) by known tridiagonal matrix algorithm and after we solve system (9) by same algorithm, using in both cases suitable boundary and initial conditions from (11), (12). In the case $\sigma \neq 0$, i.e. for solving fully implicit scheme (9)-(12) we must include a method of solving system of nonlinear algebraic equations. So, it is necessary to use Newton iterative process [19].

At the end we will note that numerical test experiments carried out on the basis on difference schemes (9)-(12) are founded on the above described algorithms.

The nonlinearities of the following kinds are considered $G(S) = \frac{1}{1+S^{\frac{1}{2}}}$. The numerical experiments are quite satisfactory and fully agree with the considered exact test solutions of problem (1)-(4). One of these solutions have the form:

$$U(x,t) = x(1-x)(1+t), \quad V(x,t) = x^2(1-x)^2\left(1+t^2\right) + 1.$$

The graphs of suitable numerical results are given in Fig. 1 and Fig. 2.

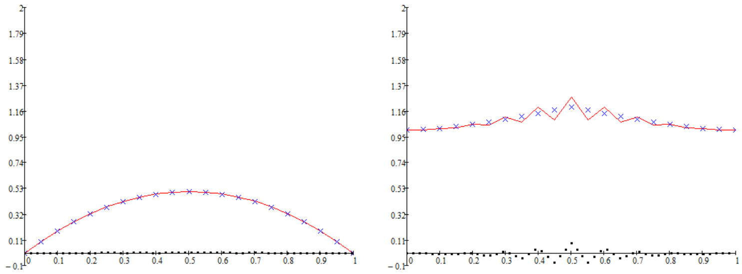

Fig. 1. Exact (solid line) and numerical (marked with $\times$) solutions U (left) and V (right) and differences between exact and numerical solutions (marked with $\bullet$) when $\varepsilon = 0.01$.

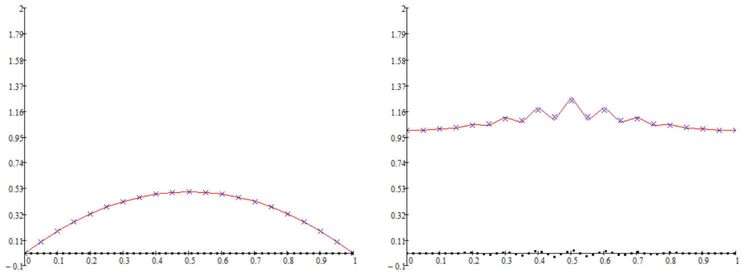

Fig. 2. Exact (solid line) and numerical (marked with $\times$) solutions U (left) and V (right) and differences between exact and numerical solutions (marked with $\bullet$) when $\varepsilon = 0.001$.

Let us note that numerical experiments give convergence of the solution u^ε, v^ε of the considered (9)-(12) schemes when $\tau \to 0$, $h \to 0$ and $\varepsilon \to 0$ to the exact solution U, V of problem (1)-(4).

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Received 12.05.2012; revised 17.09.2012; accepted 30.10.2012

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