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ON ONE NONLOCAL BOUNDARY VALUE PROBLEM OF STATICS OF THE PLANE THEORY OF ELASTIC MIXTURES

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Abstract. In the present work for two-dimensional homogeneous equations of statics of the linear theory of elastic mixture we study Bitsadze-Samarski nonlocal problem [1], in the case of finite simply-connected isotropic domain.

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Some auxiliary formulas and operators. In the two-dimensional case the basic homogeneous equations of the theory of elastic mixtures have the form [2]

$$\begin{aligned} &a_1\Delta u' + b_1 \operatorname{grad}\operatorname{div} u' + c\Delta u'' + d \operatorname{grad}\operatorname{div} u'' = 0,\\ &c\Delta u' + d \operatorname{grad}\operatorname{div} u' + a_2\Delta u'' + b_2 \operatorname{grad}\operatorname{div} u'' = 0, \end{aligned} \tag{1}$$

where $u' = (u_1, u_2)^T$ and $u'' = (u_3, u_4)^T$ are partial displacements depending on the variable x_1 and x_2; a_1, a_2, c, b_1, b_2 and d are the known constants characterizing the physical properties of the mixture and satisfying the definite conditions (inequalities) [2].

Let $D^+(D^-)$ be a finite (infinite) two-dimensional domain bounded by a smooth contour S. Suppose that $S \in C^{1,\beta}$, $0 < \beta \leq 1$, i.e. S is a Lyapunov curve.

A vector-function $u = (u_1, u_2, u_3, u_4)^T$ defined in the domain D^+ (D^-) is called regular if $u \in C^2(D^\pm) \cap C^{1,\alpha}(\overline{D^\pm})$, $0 < \alpha < \beta \leq 1$. In the case of the domain D^- we assume, in addition, the following conditions at infinity $u(x) = O(1)$, $|x|^2 \dfrac{\partial u}{\partial x_k} = O(1)$, $k = 1, 2$ to be fulfilled with $|x|^2 = x_1^2 + x_2^2$.

Note that for a regular solution of equation (1) we have the Green formula [2]

$$\int_S uNuds = \int_{D^+} N(u,u)dx, \tag{2}$$

where $N(u, u)$ is a positively defined quadratic form, the equation $N(u, u) = 0$ admits solution $u = \text{const}$; Nu is the pseudo-stress vector [2].

The following assertion holds [2].

Theorem 1. *Let $S \in C^{1,\beta}$, $0 < \beta \leq 1$ and let u be a regular solution of equation (1) in D^+. Then*

$$u(x) = \frac{1}{2\pi}\int_S ([N_y\phi(x-y)]'u^+(y) - \phi(x-y)[Nu(y)]^+)d_yS,$$

where $\phi(x-y)$ *is the basic fundamental matrix of equation* (1), $x=(x_1,x_2)$ *and* $y=(y_1,y_2)$

$$\phi(x-y)=\mathrm{Re}[m\ln(z-\zeta)+\frac{1}{4}n\frac{\overline{z}-\overline{\zeta}}{z-\zeta}],\quad z=x_1+ix_2,\quad \zeta=y_1+iy_2; \tag{3}$$

$$[N_y\phi(y-x)]'=\frac{\partial}{\partial s(y)}\mathrm{Im}\Big[E\ln(z-\zeta)-\frac{1}{2}\varepsilon\frac{\overline{z}-\overline{\zeta}}{z-\zeta}\Big]. \tag{4}$$

m, n *and* ε *are the* (4×4) *known constant matrices,* E *is the* (4×4) *unit matrix,* $\frac{\partial}{\partial s(y)}=n_1(y)\frac{\partial}{\partial y_2}-n_2(y)\frac{\partial}{\partial y_1}$, $n=(n_1,n_2)^T$ *is the unit vector of the outer normal.*

The first BVP is formulated as follows: Find a regular solution to the equations (1) in $D^{\pm}$ which satisfies the boundary condition.

$$u^{\pm}(y)=f(y),\quad y\in S, -\text{Problem}\quad (I)^{\pm}_f;$$

where f is a sufficiently smooth vector-function [2].

The following statement is valid.

Theorem 2. *Let* $s\in C^{1,\beta}$, $0<\beta\leq 1$. *Then the homogeneous problems* $(I)^{\pm}_0$ *have no nontrivial regular solutions.*

Using the way described in [3] we get

$$G(x,y,I^+)=\phi(x-y)-\frac{1}{\pi}\int_S\phi(x-\eta)g(\eta,y)d_\eta S,\quad (x,y)\in(\overline{D^+}*D^+)\backslash\Lambda$$

is the Green matrix for domain D^+ of the problem $(I)^+$, where the matrix $\phi(x-y)$ is defined by (3), $g(\eta,y)$ represents unique solution of the second kind Fredholm integral equation

$$g(t,y)+\frac{1}{\pi}\int_s N_t\phi(t-\eta)g(\eta,y)d_\eta S=N_t\phi(t-y),\quad t\in S,\quad y\in D^+;$$

$N_x\phi(x-y)$ is transpose of the matrix (4) (see. [2]), and Λ is a diagonal product $\overline{D^+}*\overline{D^+}$.

Finally, let us note that the unique solution of the $(I)^+_f$ problem ($f\in C^{1,\alpha}(S)$, $S\in C^{1,\beta}$, $0<\alpha<\beta\leq 1$) is given by the formula

$$U(x)=\frac{1}{2\pi}\int_S[N_yG(x,y,I^+)]'f(y)d_yS,\quad x\in D^+. \tag{5}$$

Statement of the problem and the method of its solving. In the known work of A. V. Bitsadze and A. A. Samarski [1], new mathematical problem with nonlocal boundary conditions are stated and studied.

In this paper of the Bitsadze-Samarski nonlocal boundary value problem for (1) equation in the finite simply-connected isotropic domain D^+ is considered. To solve the problem we use the method developed in [1] and the results given in the first section of the present work.

Let $S \in C^{1,\beta}$, $0 < \beta \leq 1$ be boundary of the domain D^+, and let Γ be a part of the S. Suppose the curve γ lies is D^+ and $\gamma = I(\Gamma)$ is a diffeomorphisim between γ and Γ.

Let us consider the following nonlocal boundary value problem: Find a regular solution of equation (1) in the D^+ which satisfies the boundary conditions

$$U(y) = F(y), \quad y \in S \backslash \Gamma; \quad U|_\gamma = U|_\Gamma; \tag{6}$$

where $F \in C^{1,\alpha}(S\backslash\Gamma)$, $0 < \alpha < \beta \leq 1$, is a given vector-function.

It is evident that problem (6) represents first generalized problem of statics in D^+.

Using the Green formula (2) it is easy to prove

Theorem 3. *Problem (6) has at most one regular solution.*

Let us prove the existence of solution of problem (6).

Suppose that $U|_\Gamma = h(y)$, ($h(y)$ is an unknown vector-function). From condition $U|_\gamma = U|_\Gamma$ we get $U|_\gamma = h(\eta)$, $\eta \in \Gamma$.

Due to formula (5) and condition (6) for determining h we obtain the following Fredholm integral equation of the second kind

$$h(\eta) - \frac{1}{2\pi}\int_\Gamma [N_y G(x,y,I^+)]'_\gamma h(y) d_y S =$$
$$= \frac{1}{2\pi}\int_{S\backslash\Gamma} [N_y G(x,y,I^+)]'_\gamma F(y) d_y S, \quad \eta \in S. \tag{7}$$

Hence problem (6) is reduced to equation (7). On the other hand, by Theorem 3 equation (7) has a unique solution.

Thus, problem (6) is solvable and the solution is representable by formula (5), where $f = F$ on $S\backslash\Gamma$ and f is a solution of equation (7) on Γ.

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