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ON ONE INTEGRAL EQUATION WITH SINGULARITY ARISING FROM TRANSPORT THEORY

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Abstract. The Chandrasekhar's equation describing the scattering of polarized light in the case of a combination of Rayleigh and isotropic scattering with arbitrary photon survival probability in an elementary scattering is considered. The Hilbert-Shmidt expansion theorem in terms of eigenvectors of discrete and continuous spectra of the corresponding characteristic equation is represented.

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Consider the vector equation of radiation transfer of the polarized light (see [1,2])

$$\mu \frac{\partial}{\partial \tau} I(\tau,\mu) + I(\tau,\mu) = \frac{1}{2}\omega Q(\mu) \int_{-1}^{+1} Q^T(\mu') I(\tau,\mu') d\mu', \tag{1}$$

$$\tau \in (-\infty, +\infty), \quad \mu \in (-1, +1),$$

where $Q(\mu)$ is the square matrix

$$Q(\mu) = \frac{3(c+2)^{1/2}}{2(c+2)} \left\| \begin{matrix} c\mu^2 + \frac{3}{2}(1-c) & (2c)^{1/2}(1-\mu^2) \\ \frac{1}{3}(c+2) & 0 \end{matrix} \right\|,$$

$\omega \in (0,1)$ is the probability in an elementary scattering, $c \in (0,1)$ is a parameter which characterize the degree of the deviation the law of scattering from the Rayleigh. The symbol T denotes the transpose. Fourier transformation of the equation (1) with respect to τ gives us the following integral equation

$$(\nu - \mu)\tilde{\psi}_\nu(\mu) = \frac{\omega\nu}{2} Q(\mu) \int_{-1}^{+1} Q^T(\mu')\tilde{\psi}_\nu(\mu') d\mu' + f(\mu), \tag{2}$$

where $f(\mu)$ is a given matrix function and ν is a parameter. It is seen that, if $f(\mu) = 0$, then we obtain corresponding of(1) characteristic equation, corresponding to (1)

$$(\nu - \mu)\psi_\nu(\mu) = \frac{\omega\nu}{2} Q(\mu) \int_{-1}^{+1} Q^T(\mu')\psi_\nu(\mu') d\mu'. \tag{3}$$

The values of ν for which (3) has nonzero solutions, are the eigenvalues of the equation. The set of all eigenvalues will be denoted by $S[\nu]$. The discrete spectrum of (3) consists of two real points $\{\pm\nu_0\}$, which correspond to the two eigenfunctions (see e.g. [2])

$$\psi_{\pm\nu_0}(\mu) = \left\| \begin{matrix} \psi^{(1)}_{\pm\nu_0}(\mu) \\ \psi^{(2)}_{\pm\nu_0}(\mu) \end{matrix} \right\|.$$

There a continuum of values of ν, namely $-1 \le \nu \le 1$, for which (3) has a solution in the distributional sense: (cf.[3])

$$\psi_\nu(\mu) = \frac{\omega\nu}{2}(\nu-\mu)^{-1}M(\nu,\mu) + \delta(\nu-\mu)\Big(E - \frac{\omega\nu}{2}\int_{-1}^{+1}(\nu-\mu')^{-1}M(\nu,\mu')d\mu'\Big),$$

where

$$M(\nu,\mu) = Q(\mu)Q^T(\nu) + \frac{\omega\nu}{2}Q(\mu)K(\nu)Q^T(\nu)\Big(E - \frac{\omega\nu}{2}K(\nu)\Big)^{-1},$$

$$K(\nu) = \int_{-1}^{+1}(Q^T(\mu') - Q^T(\nu))(\nu-\mu')^{-1}Q(\mu')d\mu',$$

E is the unit matrix and δ is the Dirac function.

The eigenfunctions of (3) obeys the following orthogonality condition

$$\int_{-1}^{+1}\mu\psi_\nu(\mu)\psi_{\nu'}(\mu)d\mu = D(\nu,\nu')N(\nu),$$

where

$$D(\nu,\nu') = \begin{cases} 0, & when\ \ \nu \ne \nu', \\ \delta(\nu-\nu'), & when\ either\ \ \nu\ \ or\ \ \nu'\ \ are\ \ continuous, \\ \delta_{ij}, & when\ \ both\ \ \nu=\nu_i,\ \ \nu'=\nu_j\ \ are\ \ discrete. \end{cases}$$

$N(\nu)$ is the normalization matrix and δ_{ij} is the Kronecker symbol.

The set of eigenfunctions is complete and hence it obeys the relation

$$\mu\int_{S[\nu]}\psi_\nu(\mu)d\Gamma(\nu)\psi_\nu(\mu') = \delta(\mu-\mu')E,$$

where the integral on the left-hand side is the spectral integral and

$$d\Gamma(\nu) = \begin{cases} N^{-1}(\nu)d\nu, & when\ \ \nu\ \ is\ a\ continuum\ eigenvalue, \\ \sum_j \frac{\delta(\nu-\nu_j)}{N(\nu_j)}d\nu, & when\ \ \nu\ \ is\ not\ a\ continuum\ eigenvalue, \end{cases}$$

here, the sum on the right-hand side is over all discrete eigenvalues.

For the equation (2) we can prove the following statements.

Theorem 1. *If* $\nu\bar{\in}S[t]$, *then equation (2) has a unique solution* $\tilde{\psi}_\nu \in H^*$ *for any* $f(\mu)$. *The solution of this equation is given by formula*

$$\tilde{\psi}_\nu(\mu) = \int_{S[t]}\frac{t}{t-\nu}\psi_t(\mu)d\Gamma(t)\int_{-1}^{+1}\psi_t(\mu')f(\mu')d\mu'. \tag{4}$$

Theorem 2. *Let* $\nu=\nu_0$ *be an eigenvalue of (3). Then equation (2) is solvable, if and only if the function* f *satisfies the condition*

$$\int_{-1}^{+1}\psi_{\nu_0}(\mu)f(\mu)d\mu = 0.$$

Provided these conditions are satisfied, then solution of (2) may be written as

$$\tilde{\psi}_{\nu_0}(\mu) = c_0\psi_{\nu_0}(\mu) - \frac{1}{2}N^{-1}\psi_{-\nu_0}$$

$$+\int_{-1}^{+1} \frac{t}{t-\nu}\psi_t(\mu)(\mu)d\Gamma(t)\int_{-1}^{+1}\psi_t(\mu')f(\mu')d\mu',$$

where c_0 *is an arbitrary constant.*

Theorem 3. *If* $\nu = t_0$ *where* $t_0 \in]-1,+1[$. *Then equation (2) is solvable, if and only if the function* f *satisfies the condition*

$$\int_{-1}^{+1}\psi_\nu(\mu')f(\mu')d\mu' = 0.$$

Provided these conditions are satisfied, equation (2) has one and only one solution and this solution may be given by (4).

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