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ON THE APPROXIMATE SOLUTION OF A NONLINEAR SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

Peradze J.

Abstract. A system of equations for the static Timoshenko beam is solved using the proposed iteration method. The method error is estimated.

Keywords and phrases: Timoshenko beam equation, iteration method, Chipot's method, algorithm error.

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Statement of the problem. Consider the following boundary value problem

$$\begin{gathered}\left(cd-a+b\int_0^1 w'^2(x)\,dx\right)w''(x)-cd\psi'(x)=f(x),\\ \psi''(x)-cd\,(\psi(x)-w'(x))=g(x),\end{gathered}\tag{1}$$

$$0<x<1,$$

$$w(0)=w(1)=0,\quad \psi'(0)=\psi'(1)=0,\tag{2}$$

where $f(x)$ and $g(x)$ are given continuous functions, $w(x)$ and $\psi(x)$ are the sought solutions, a, b, c and d are given positive constants, $cd-a>0$.

System (1) is a static problem for the Timoshenko beam [5]–[7]. In the present paper, an iteration method of constructing an approximate solution of problem (1),(2) is proposed and the condition for its convergence is established. As far as we know, no algorithm analogous to the one considered below was applied previously for system (1).

Reduction of the problem. From the second equation of system (1) we find

$$\psi(x)=\int_0^1 R(x,\xi)w(\xi)\,d\xi+\int_0^1 S(x,\xi)g(\xi)\,d\xi,\tag{3}$$

where

$$R(x,\xi)=-\frac{cd}{\sinh(\sqrt{cd})}\begin{cases}\cosh\left(\sqrt{cd}\,(x-1)\right)\sinh\left(\sqrt{cd}\,\xi\right), & 0<\xi<x<1,\\ \cosh\left(\sqrt{cd}\,x\right)\sinh\left(\sqrt{cd}\,(\xi-1)\right), & 0<x<\xi<1,\end{cases}$$

$$S(x,\xi)=-\frac{1}{\sqrt{cd}\sinh(\sqrt{cd})}\begin{cases}\cosh\left(\sqrt{cd}\,(x-1)\right)\cosh\left(\sqrt{cd}\,\xi\right), & 0<\xi<x<1,\\ \cosh\left(\sqrt{cd}\,x\right)\cosh\left(\sqrt{cd}\,(\xi-1)\right), & 0<x<\xi<1.\end{cases}$$

Using (3) in the first equation of system (1) and taking into consideration (2), we obtain

$$\left(cd - a + b\int_0^1 w'^2(x)\,dx\right)w''(x) + \int_0^1 G(x,\xi)w'(\xi)\,d\xi = p(x), \tag{4}$$

$$w(0) = w(1) = 0. \tag{5}$$

Here

$$G(x,\xi) = -\frac{(cd)^2}{\sinh(\sqrt{cd})}\begin{cases}\sinh\left(\sqrt{cd}\,(x-1)\right)\cosh\left(\sqrt{cd}\,\xi\right), & 0<\xi<x<1,\\ \sinh\left(\sqrt{cd}\,x\right)\cosh\left(\sqrt{cd}\,(\xi-1)\right), & 0<x<\xi<1,\end{cases}$$

$$p(x) = f(x) + \frac{1}{cd}\int_0^1 G(x,\xi)g(\xi)\,d\xi. \tag{6}$$

System (4),(5) is a problem relative to $w(x)$ whose solution found by means of (3) gives $\psi(x)$.

Algorithm. We will solve problem (4),(5) by iteration. After choosing as the initial approximation some function $w_0(x)$ that vanishes at the points $x=0$ and $x=1$, we will define the next approximations by the formula

$$\left(cd - a + b\int_0^1 w_k'^2(x)\,dx\right)w_k''(x) + \int_0^1 G(x,\xi)w_{k-1}'(\xi)\,d\xi = p(x), \tag{7}$$
$$k = 1,2,\dots\,,$$

with the condition

$$w_k(0) = w_k(1) = 0, \tag{8}$$

where $w_{k+l}(x)$ is the k-th iterative approximation of the solution of problem (4), (5), $k=1,2,\dots,\quad l=-1,0$.

For $w_k(x)$, system (7),(8) is a nonlinear problem. To solve it, we rewrite (7) in the form

$$\left(cd - a + b\int_0^1 w_k'^2(x)\,dx\right)w_k''(x) = p_{k-1}(x), \tag{9}$$

where

$$p_{k-1}(x) = p(x) - \int\limits_0^1 G(x,\xi)w_{k-1}'(\xi)\,d\xi,$$

and seek $w_k(x)$ by Chipot's method [1], [2] in the form of product

$$w_k(x) = \lambda_k v_k(x). \tag{10}$$

Here the function $v_k(x)$ and the parameter λ_k are the respective solutions of the boundary value problem

$$v_k''(x) = p_{k-1}(x),$$

$$v_k(0) = v_k(1) = 0,$$

and the cubic equation

$$\lambda_k(cd - a + bs_k\lambda_k^2) - 1 = 0$$

in which

$$s_k = \int_0^1 v_k'^2(x)\,dx,$$

and which has a unique real solution. The latter fact is confirmed by means of the function $\gamma(\lambda) = \lambda(c_1 + c_2\lambda^2) - c_3,\ \ c_1, c_2, c_3 > 0$.

Using the well-known relations, we first find the function $v_k(x)$ by the formula

$$v_k(x) = \lambda_{k-1}\int_0^1 H(x,\xi)\int_0^1 G(\xi,\eta)v_{k-1}'(\eta)\,d\eta\,d\xi + r(x),$$

where

$$H(x,\xi) = -\begin{cases}(x-1)\xi, & 0<\xi<x<1,\\ x(\xi-1), & 0<x<\xi<1,\end{cases}$$

$$r(x) = -\int_0^1 H(x,\xi)p(\xi)\,d\xi,$$

and then, having preliminarily calculated the constant s_k, by virtue of the equality

$$\lambda_k = \left(\frac{1}{2bs_k}\right)^{\frac{1}{3}}\sum_{l=1}^{2}\left(1 + (-1)^l\left(1 + \frac{4}{27}\,\frac{(cd-a)^3}{bs_k}\right)^{\frac{1}{2}}\right)^{\frac{1}{3}}$$

we obtain the value of the parameter λ_k.

The substitution of λ_k and $v_k(x)$ in (10) gives the k-th iterative approximation $w_k(x)$.

Error of the algorithm. Under the error of the algorithm we will understand a difference $w_k(x) - w(x)$ where $w_k(x)$ is the k-th iterative approximation of process (7), and $w(x)$ is an exact solution of problem (4), (5). The following statement is valid

Theorem. *If the condition $q < 1$, where*

$$q = \frac{1}{\sqrt{2}\,(cd-a)}\left(\int_0^1\int_0^1 G^2(x,\xi)\,dx\,d\xi\right)^{\frac{1}{2}}$$

$$= \frac{(cd)^2}{2\sqrt{2}\,(cd-a)\sinh(\sqrt{cd})}\left(\frac{\sinh(2\sqrt{cd})}{2\sqrt{cd}} - 1\right)^{\frac{1}{2}},$$

is fulfilled, then $w_k(x) \to w(x)$ in the norm of the space $L_2(0,1)$, and for the error we have the estimate

$$\left\|\frac{d^l}{dx^l}\,(w_k(x) - w(x))\right\|_{L_2(0,1)} \le \left(\frac{1}{2}\right)^{\frac{1-l}{2}} q^k\|w_0'(x) - w'(x)\|_{L_2(0,1)}\,,$$

$$k = 1,2,\dots\,, \quad l = 0,1.$$

Let us give some values of the parameter q. If $cd = 0.1$, $a = 0.05$, then $q = 0.057$, if $cd = 1$, $a = 0.2$, then $q = 0.3392$, if $cd = 3$, $a = 0.6$, then $q = 0.9197$.

The question of iteration process convergence for the axially-symmetric Timoshenko static plate is considered in [3] and [4].

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Author's addresses:

J. Peradze
Iv. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia

Georgian Technical University
77, M. Kostava St., Tbilisi 0175
Georgia
E-mail: j_peradze@yahoo.com