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FINITE DIFFERENCE SCHEME AND STABILITY OF THE STATIONARY SOLUTION FOR ONE SYSTEM OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. One-dimensional analog of the system of nonlinear partial differential equations arising in process of vein formation of young leaves is considered. Finite difference scheme for initial-boundary value problem for this system is studied. Stability of the stationary solution is confirmed by the constructed iterative algorithm.

Keywords and phrases: One-dimensional system of nonlinear partial differential equations, vein formation model, finite difference scheme, iterative algorithm, stability of the stationary solution.

AMS subject classification: 35Q92, 65N06.

One type two-dimensional model of describing the vein formation of young leaves is stated in [1]. This model has the form

$$\begin{gathered}\frac{\partial S}{\partial t}=\frac{\partial}{\partial x_1}\left(D_1\frac{\partial S}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(D_2\frac{\partial S}{\partial x_2}\right),\\ \frac{\partial D_i}{\partial t}=f\left(D_i,D_i\frac{\partial S}{\partial x_i}\right),\quad i=1,2,\end{gathered}\tag{1}$$

where $S(t,x_1,x_2)$ is a signal concentration, D_1 and D_2 are diffusion coefficients with directions of OX_1 and OX_2 axis, respectively.

The following initial boundary value problem for one-dimensional analog of system (1) is investigated in [2]:

$$\begin{gathered}\frac{\partial S}{\partial t}=\frac{\partial}{\partial x}\left(D\frac{\partial S}{\partial x}\right),\quad (x,t)\in(0,1)\times(0,T],\\ \frac{\partial D}{\partial t}=-D+g\left(D\frac{\partial S}{\partial x}\right),\quad (x,t)\in[0,1]\times(0,T],\\ S(0,t)=0,\quad D\frac{\partial S}{\partial x}\bigg|_{x=1}=\psi,\quad t\in[0,T],\\ S(x,0)=S_0(x);\quad D(x,0)=D_0(x)\geq\delta_0=const>0,\ x\in[0,1],\\ 0<g_0\leq g(\xi)\leq G_0,\quad g_0=Const,\quad G_0=Const,\end{gathered}\tag{2}$$

where T and ψ are positive constants, g, S_0, D_0 are given sufficiently smooth functions. In [2] the authors proved are uniqueness and existence theorem of problem (2). They have used the new designation and iterative algorithm.

In particular, with introducing the new unknown function

$$W(x,t) = D\frac{\partial S}{\partial x}$$

problem (2) was reduced to the following form

$$\begin{gathered}
\frac{\partial W}{\partial t} = D\frac{\partial^2 W}{\partial x^2} + \left[\frac{g(W)}{D} - 1\right] W, \\
\frac{\partial D}{\partial t} = -D + g(W), \\
\left.\frac{\partial W}{\partial x}\right|_{x=0} = 0, \quad W(1,t) = \psi, \\
W(x,0) = W_0(x) = D_0(x)\frac{dS_0(x)}{dx}, \quad d(x,0) = D_0(x).
\end{gathered} \tag{3}$$

The corresponding iterative scheme looks like:

$$\begin{gathered}
\frac{\partial W^n}{\partial t} = D^{n-1}\frac{\partial^2 W^n}{\partial x^2} + \left[\frac{g\left(W^{n-1}\right)}{D^{n-1}} - 1\right] W^{n-1}, \\
\frac{\partial D^n}{\partial t} = -D^n + g\left(W^{n-1}\right), \\
\left.\frac{\partial W^n}{\partial x}\right|_{x=0} = 0, \quad W^n(1,t) = \psi, \\
W^n(x,0) = W_0(x), \quad D^n(x,0) = D_0(x),
\end{gathered} \tag{4}$$

where by n the number of iteration is denoted.

In the same work [2] under some restrictions on the given data global asymptotic stability of the stationary solution of problem (2) is established. Hopf type bifurcation is also studied. In particular, if the following restrictions $0 < \psi < \psi_c$ take place, where ψ_c is some critical value, then stationary solution of problem (2) is stable and as $\psi > \psi_c$, then it becomes unstable and Hopf bifurcation occurs.

Many scientific works are devoted to investigation of initial-boundary value problems for (1), (2) models and construction of numerical resolution algorithms as well (see, for example, [3]-[15] and references therein). In the direction of biological modeling it is necessary to note the work [10], where many mathematical models of similar diffusion processes are also presented and discussed.

Let us also note that in [14] the semi-discrete and implicit finite difference schemes are constructed and investigated for problem (2).

The aim of our note, as it was remarked above, was by using (4) iterative algorithm to construct the numerical resolution scheme and to establish stability of the stationary solution of problem (3) by the numerical experiments.

Let $\omega_{h\tau}$ be the uniform grid on the rectangle $[0,1] \times [0,T]$ with the steps h and τ. With the help of direct method of construction of discrete models (see, for example, [16]) and using usual notations let us consider the following difference scheme with the above iterative algorithm as well:

$$w_t^n = d^{n-1} w_{\bar{x}x}^n + \left[\frac{g\left(w^{n-1}\right)}{d^{n-1}} - 1\right] w^{n-1},$$

$$d_t^n = -d^n + g\left(w^{n-1}\right), \tag{5}$$

$$w_{x,0}^n = 0, \quad w_M^n = \psi,$$

$$w^n\left.\right|_{t-0} = W_0(x), \quad d^n\left.\right|_{t=0} = D_0(x).$$

It is clear that the boundary condition of the scheme (5) on the left side has only first rate, but it is not difficult to build second rate condition as well. Such approximation for problem (2) in note [14] is given, for example.

The (5) iterative algorithm is easily realized on the computer because it contains two independent linear models. In each time step at first we find d from second equation of system (5) and after vector w solving first linear system with tridiagonal matrix.

The following statement takes place.

Theorem. *Solution w^n, d^n of problem (5) converges to the solution W, D of problem (3) in all grid points then $\tau \to 0$, $h \to 0$ and $n \to \infty$.*

Using described algorithm (5) and discrete analogs investigated in [14] many numerical experiments are carried out. These experiments agree with theoretical results of stability of the stationary solution of problem (2).

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