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ENSURING THE BOUNDARY CONDITION OF FACE SURFACES FOR NON-SHALLOW SHELLS

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Abstract. I. Vekua suggested a simple method ensuring the boundary conditions of the face surfaces for shallow shells. In this paper the result is generalized for non-shallow shells.

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1. Using the vector and tensor notations, the equilibrium equation of the continuous medium and the stress-strain relations (Hook's Law) can be written in the form (see [1]):

$$\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}\boldsymbol{\tau}^{\boldsymbol{i}}}{\partial x^i}+\boldsymbol{\Psi}=0,\quad \hat{\nabla}_i\boldsymbol{\tau}^{\boldsymbol{i}}+\boldsymbol{\Psi}=0, \tag{1}$$

$$\boldsymbol{\tau}^{\boldsymbol{i}}=\hat{\mathbb{C}}^{ij}\partial_j\boldsymbol{U},\quad \hat{\mathbb{C}}^{ij}=\lambda\mathbf{R}^i\otimes\mathbf{R}^j+\mu\mathbf{R}^j\otimes\mathbf{R}^i+\mu(\mathbf{R}^i\cdot\mathbf{R}^j)E, \tag{2}$$

$$(E=\mathbf{R}^i\otimes\mathbf{R}_i,\quad i,j=1,2,3),$$

where g is the discriminant of the metric quadratic form of the space, $\boldsymbol{\tau}^{\boldsymbol{i}}$ are contravariant constituents of the stress vector, $\boldsymbol{\Psi}$ is the volume force, $\hat{\nabla}_i$ are covariant derivatives with the space coordinates x^i, $\boldsymbol{U}$ is the displacement vector, λ and μ are Lame's constants, $\otimes$ denotes the tensor product, $\mathbf{R}_i=\partial_i\mathbf{R}$ and $\mathbf{R}^i$ are covariant and contravariant basis vectors of the curvilinear coordinate system x^1,x^2,x^3 moreover

$$\mathbf{R}_i\cdot\mathbf{R}^j=\delta_i^j,\quad g_{ij}=\mathbf{R}_i\cdot\mathbf{R}_j,\quad g^{ij}=\mathbf{R}^i\cdot\mathbf{R}^j,\quad \mathbf{R}^i=g^{ij}\mathbf{R}_j.$$

For the 3-D shell-type elastic bodies Ω is more convenient to consider the coordinate system which is normally connected with the midsurface S. This means that the radius-vector $\boldsymbol{R}$ of any point of the domain Ω can be represented in the form

$$\boldsymbol{R}(x^1,x^2,x^3)=\boldsymbol{r}(x^1,x^2)+x^3\boldsymbol{n}(x^1,x^2),\quad -h\le x^3\le h,$$

where $\boldsymbol{r}$ and $\boldsymbol{n}$ are respectively the radius vector and the unit vector of the normal of the mudsurface $S(x^3=0)$ and x^1,x^2 are the Gaussian parameters of the S, h is semi-thickness of shall Ω.

The covariant and contravariant basis vectors $\mathbf{R}_i$ and $\mathbf{R}^i$ of the surface $\hat{S}(x^3=const)$ and the corresponding bases vector $\mathbf{r}_i$ and $\mathbf{r}^i$ of the midsurface $S(x^3=0)$ are connected by the following relations:

$$\begin{gathered}\boldsymbol{R}_\alpha=(a_\alpha^{\alpha_1}-x_3b_\alpha^{\alpha_1})\boldsymbol{r}_{\alpha_1}\quad \boldsymbol{R}^\alpha=\frac{a_\alpha^{\alpha_1}+x_3(b_\alpha^{\alpha_1}-2Ha_\alpha^{\alpha_1})}{1-2Hx_3+Kx_3^2}\boldsymbol{r}^{\alpha_1},\quad \boldsymbol{R}_3=\boldsymbol{r}_3=\boldsymbol{n},\\ a_\alpha^{\alpha_1}=\boldsymbol{r}_\alpha\boldsymbol{r}^{\alpha_1},\quad \boldsymbol{r}_\alpha=\partial_\alpha\boldsymbol{r},\quad b_\alpha^{\alpha_1}=-\boldsymbol{n}_\alpha\boldsymbol{r}^{\alpha_1},\quad 2H=b_\alpha^\alpha,\quad K=b_1^1b_2^2-b_2^1b_1^2,\end{gathered} \tag{3}$$

$$(x^3 = x_3), \quad (\alpha, \alpha_1 = 1, 2).$$

By shallow shells I. Vekua meant 3-D shell-type elastic bodies satisfying the following conditions:

$$\boldsymbol{R}_\alpha \cong \boldsymbol{r}_\alpha, \quad \boldsymbol{R}^\alpha \cong \boldsymbol{r}^\alpha, \quad (\boldsymbol{R}^3 = \boldsymbol{R}_3 = \boldsymbol{r}^3 = \boldsymbol{r}_3 = \boldsymbol{n}).$$

In the sequel, under non-shallow shells we mean elastic bodies from which $\mathbf{R}_\alpha$ and $\mathbf{R}^\alpha$ have the form (3).

2. There are many different methods of reducing 3-D problem of theory of elasticity to 2-D ones of the theory of shells. In the present paper we realize the reduction by the method suggested by I. Vekua. Since the system of Legendre polynomials $P_m\left(\frac{x^3}{h}\right)$ is complete in the interval $[-1, 1]$ for (1) and (2) we obtain the equivalent infinite system of 2-D equations

$$\int_{-h}^{h} \left[\frac{1}{\sqrt{a}} \frac{\partial \sqrt{a} \vartheta \boldsymbol{\tau}^{\boldsymbol{\alpha}}}{\partial x^\alpha} + \frac{\partial \vartheta \boldsymbol{\tau}^{\boldsymbol{3}}}{\partial x^3} + \vartheta \boldsymbol{\Psi}\right] P_m\left(\frac{x^3}{h}\right) dx^3,$$

$$(-h \leq x_3 \leq h, \quad \vartheta = 1 - 2Hx_3 + Kx_3^2)$$

or in the form

$$\nabla_\alpha \overset{(m)}{\boldsymbol{\sigma}}{}^{\alpha} - \overset{(m)}{\underline{\boldsymbol{\sigma}}}{}^3 + \overset{(m)}{\boldsymbol{\Phi}} + \frac{2m+1}{2h}\left[\overset{(+)}{\vartheta} \overset{(+)}{\boldsymbol{\tau}}{}^3 - (-1)^m \overset{(-)}{\vartheta} \overset{(-)}{\boldsymbol{\tau}}{}^3\right] = 0, \tag{4}$$

$$(m = 0, 1, ...; \quad \overset{(\pm)}{\vartheta} = 1 \mp 2Hh + Kx_h^2),$$

where ∇_α are covariant derivatives with the parameters x^1, x^2 and

$$\overset{(m)}{\boldsymbol{\sigma}}{}^{i} = \frac{2m+1}{2h} \int_{-h}^{h} \vartheta \boldsymbol{\tau}^i P_m\left(\tfrac{x^3}{h}\right) dx_3 = \sum_{s=0}^{\infty} \overset{(m)}{A}{}^{ij}_{i_1 j_1} \mathbb{C}^{i_1 j_1} D_j \overset{(s)}{\boldsymbol{U}},$$

$$\overset{(m)}{\underline{\boldsymbol{\sigma}}}_3 = \frac{2m+1}{h}\left(\overset{(m-1)}{\boldsymbol{\sigma}_3} + \overset{(m-3)}{\boldsymbol{\sigma}_3} + \cdots\right).$$

$$\left(\overset{(m)}{\boldsymbol{U}}, \overset{(m)}{\boldsymbol{\Phi}}\right) = \frac{2m+1}{2h} \int_{-h}^{h} (\boldsymbol{U}, \vartheta \boldsymbol{\Psi}) P_m\left(\tfrac{x^3}{h}\right) dx_3, \quad \overset{(\pm)}{\boldsymbol{\tau}}{}^3 = \boldsymbol{\tau}^3(x^1, x^2, \pm h),$$

$$\underset{(s)}{\overset{(m)}{A}}{}^{ij}_{i_1 j_1} = \frac{2m+1}{2h} \int_{-h}^{h} \vartheta A^i_{i_1} A^j_{j_1} P_s\left(\frac{x_3}{h}\right) P_m\left(\frac{x_3}{h}\right) dx_3, \tag{5}$$

$$A^\alpha_{\alpha_1} = \vartheta^{-1}[a^\alpha_{\alpha_1} + x_3(b^\alpha_{\alpha_1} - 2Ha^\alpha_{\alpha_1})], \quad A^i_3 = A^3_i = \delta_{i3},$$

$$\mathbb{C}^{i_1 j_1} = \lambda \mathbf{r}^{i_1} \otimes \mathbf{r}^{j_1} + \mu \mathbf{r}^{j_1} \otimes \mathbf{r}^{i_1} + \mu(\mathbf{r}^{i_1} \cdot \mathbf{r}^{j_1})\mathbb{E}, \quad \mathbb{E} = \boldsymbol{r}^\alpha \otimes \boldsymbol{r}_\alpha + \boldsymbol{n} \otimes \boldsymbol{n},$$

$$D_\alpha \overset{(s)}{\boldsymbol{U}} = \partial_\alpha \overset{(s)}{\boldsymbol{U}}, \quad D_3 \overset{(s)}{\boldsymbol{U}} = \frac{2s+1}{h} \left(\overset{(s+1)}{\boldsymbol{U}} + \overset{(s+3)}{\boldsymbol{U}} + \cdots \right).$$

Consider the first $N+1$ equations (4) and (5), and assume that $\overset{(m)}{\boldsymbol{U}} = 0$ if $m > N$, where N is non-negative integer (approximation of order N). Then we shall have

$$\boldsymbol{U}_N(x^1, x^2, x^3) = \sum_{m=0}^{N} \overset{(m)}{\boldsymbol{U}}(x^1, x^2) P_m \left(\frac{x_3}{h} \right),$$

$$\boldsymbol{\sigma}^i_N(x^1, x^2, x^3) = \vartheta \boldsymbol{\tau}^i_N = \sum_{m=0}^{N} \overset{(m)}{\boldsymbol{\sigma}}(x^1, x^2) P_m \left(\frac{x_3}{h} \right) = \boldsymbol{\sigma}^i(\boldsymbol{U}_N).$$

There expressions satisfy boundary conditions on lateral surfaces Σ of a shell Ω, but the boundary conditions on face surfaces $x^3 = \pm h$

$$\overset{(\pm)}{\boldsymbol{\sigma}}{}^3 = \overset{(\pm)}{\vartheta} \overset{(\pm)}{\boldsymbol{\tau}}{}^3 = \overset{(\pm)}{\vartheta} \overset{(\pm)}{\boldsymbol{q}}$$

are not, in general, satisfied.

Now arises the problem: To find vector $\boldsymbol{V}(x^1, x^2, x^3)$ satisfying the following conditions:

1. $\overset{(\pm)}{\boldsymbol{\tau}}{}^3 = \overset{(\pm)}{\boldsymbol{q}}$, $(x^3 = \pm h)$;
2. $\overset{(m)}{\boldsymbol{V}} = 0$, $\overset{(m)}{\boldsymbol{\sigma}}{}^i(\boldsymbol{U}_N + \boldsymbol{V}) = 0$, $m \le N$;
3. $\forall \varepsilon$, $\exists \delta > 0 \Rightarrow |\boldsymbol{V}(x^1, x^2, x^3)| < \varepsilon$ and $|\boldsymbol{\sigma}^i(\boldsymbol{V})| < \varepsilon$, if $-h + \delta \le x^3 \le h - \delta$.

For shallow shell this vector is constructed by I. Vekua. For non-shallow shell analogous vector has the form

$$\boldsymbol{V} = \vartheta^2 \left[\boldsymbol{M}_k \left(P_{k+2} \frac{x^3}{h} - P_k \frac{x^3}{h} \right) + \boldsymbol{M}_{k+1} \left(P_{k+3} \frac{x^3}{h} - P_{k+1} \frac{x^3}{h} \right) \right],$$

where $k > N + 5$ and

$$\boldsymbol{M}_k = \frac{h}{2} \frac{\mathbb{C}^{-1}_{33}}{2k+3} \left(\frac{\overset{(+)}{\vartheta} \overset{(+)}{\boldsymbol{q}} - \overset{(+)}{\boldsymbol{\sigma}}{}^3_N}{\overset{(+)}{\vartheta}{}^3} - (-1)^k \frac{\overset{(-)}{\vartheta} \overset{(-)}{\boldsymbol{q}} - \overset{(-)}{\boldsymbol{\sigma}}{}^3_N}{\overset{(-)}{\vartheta}{}^3} \right),$$

$$\mathbb{C}^{-1}_{33} = \frac{1}{\mu} \boldsymbol{r}^\alpha \otimes \boldsymbol{r}_\alpha + \frac{1}{\lambda + 2\mu} \boldsymbol{n} \otimes \boldsymbol{n}.$$

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