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NUMERICAL REALIZATION OF BOUNDARY VALUE PROBLEMS FOR ORDINARY INTEGRO-DIFFERENTIAL EQUATIONS BY ALTERNATING TO PERTURBATION TECHNIQUE METHOD

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Abstract. In this work we consider the problems connected with numerical realization of the alternating to perturbation technique method for boundary value problems of ordinary second order linear integro-differential equation. The use of numerical processes caused the error between exact and approximate solutions for which is getting an estimate. Then by this procedure are solved approximately some examples with known exact solutions by which it is possible to investigate the influences of round-of errors and numerical methods.

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Let us consider the non-homogenous operator equation

$$Lu + \varepsilon Mu = f, \tag{1}$$

where L, M are linear operators defining in the corresponding normalized space and there exist the following inverse operators L^{-1} and $(L+\varepsilon M)^{-1}$ parameter $\varepsilon \in [-1,1]$. For solution of (1) we used the following expression

$$u(x) = \gamma \sum_{k=0}^{\infty} \varepsilon^k \nu_k(x) + (1-\gamma) \sum_{k=0}^{\infty} P_k(\varepsilon) w_k(x), \tag{2}$$

where $\{P_k(\varepsilon)\}$ is Legendre polynomial system, $\nu_k(x)$ and $w_k(x)$ are unknown coefficients, γ is the arbitrary parameter.

As it's well known when $\gamma = 1$ by series (2) it's possible from (1) to define the explicit processes which are named as the perturbation technique or asymptotic method of solution of equation (1). The difficulties are related with the numerical realizations and other applications of asymptotic methods investigated by many authors (see i.e. [1-3]).

Below we consider the case when $\gamma = 0$. We have

$$u(x) = \sum_{k=0}^{\infty} P_k(\varepsilon) w_k(x), \quad \varepsilon \in [-1,1]. \tag{3}$$

Then from (1) it follows:

$$Lw_0(x) \cdot P_0(\varepsilon) + Lw_1(x) \cdot P_1(\varepsilon) + \cdots +$$
$$+Mw_0(x) \cdot \varepsilon P_0(\varepsilon) + Mw_1(x) \cdot \varepsilon P_1(\varepsilon) + \cdots = f(x)$$

or, using the well-known identity for Legendre polynomials

$$\varepsilon P_n(\varepsilon) = \frac{n}{2n+1} P_{n-1}(\varepsilon) + \frac{n+1}{2n+1} P_{n+1}(\varepsilon)$$

we have (for details see [4; 5]):

$$Lw_0(x) \cdot P_0(\varepsilon) + Lw_1(x) \cdot P_1(\varepsilon) + Lw_2(x) \cdot P_2(\varepsilon) + \cdots + Mw_0(x) \cdot P_1(\varepsilon)$$

$$+Mw_1(x) \cdot \left(\frac{1}{3}P_0(\varepsilon) + \frac{2}{3}P_2(\varepsilon)\right) + Mw_2(x)\left(\frac{2}{5}P_1(\varepsilon) + \frac{3}{5}P_2(\varepsilon)\right) + \cdots = f(x),$$

$$\begin{cases} Lw_0 + \dfrac{1}{3}Mw_1 = f(x), \\ Lw_k(x) + \dfrac{k}{2k-1}Mw_{k-1}(x) + \dfrac{k+1}{2k+3}Mw_{k+1}(x) = 0, \quad k \geq 1. \end{cases} \tag{4}$$

If we denote $b_k \equiv Lw_k(x)$; $t_k \equiv Mw_k(x)$; $k = 0; 1; 2; 3; \cdots$, we have:

$$\begin{cases} b_0 + \dfrac{1}{3}t_1 = f, \\ b_k + \dfrac{k}{2k-1}t_{k-1} + \dfrac{k+1}{2k+3}t_{k+1} = 0; \quad k = 1; 2; 3; \cdots . \end{cases} \tag{4a}$$

Instead of (4a) we choose the finite number of operator equations: $k = 0, 1, \cdots, 2n$. These systems split into two subsystems:

b_{2k}, t_{2k+1}	b_{2k}, t_{2k+1}
$b_0 + \frac{1}{3}t_1 = f$	$b_1 + t_0 + \frac{2}{5}t_2 = 0$
$b_2 + \frac{2}{3}t_1 + \frac{3}{7}t_2 = 0$	$b_3 + \frac{3}{5}t_2 + \frac{4}{9}t_4 = 0$
....	....
$b_{2k} + \frac{2k}{4k-1}t_{2k-1} + \frac{2k+1}{4k+3}t_{2k+1} = 0$	$b_{2k-1} + \frac{2k-1}{4k-3}t_{2k-2} + \frac{2k}{4k+1}t_{2k} = 0$
....	....
$b_{2n-2} + \frac{2n-2}{4n-5}t_{2n-3} + \frac{2n-1}{4n-1}t_{2n-1} = 0$	$b_{2n-3} + \frac{2n-3}{4n-7}t_{2n-4} + \frac{2n-2}{4n-3}t_{2n-2} = 0$
$b_{2n} + \frac{2n}{4n-1}t_{2n-1} = 0;$	$b_{2n-1} + \frac{2n-1}{4n-3}t_{2n-2} = 0.$

For $n = 0, 1, 2, 3$ we have:

$n=0$	$n=1$	$n=2$	$n=3$
$w_0 = \psi_1$	$w_0 = \psi_1 + \frac{1}{3}\psi_5$ $w_1 = -\psi_3$ $w_2 = \frac{2}{3}\psi_5$	$w_0 = \psi_1 + \frac{1}{3}\psi_5 + \frac{1}{5}\psi_9$ $w_1 = -\psi_3 - \frac{3}{5}\psi_7$ $w_2 = \frac{2}{3}\psi_5 + \frac{4}{7}\psi_9$ $w_3 = -\frac{2}{5}\psi_7$ $w_4 = \frac{8}{35}\psi_9$	$w_0 = \psi_1 + \frac{1}{3}\psi_5 + \frac{1}{5}\psi_9 + \frac{1}{7}\psi_{13}$ $w_1 = -\psi_3 - \frac{3}{5}\psi_7 - \frac{3}{7}\psi_{11}$ $w_2 = \frac{2}{3}\psi_5 + \frac{4}{7}\psi_9$ $w_3 = -\frac{2}{5}\psi_7 - \frac{4}{9}\psi_{11}$ $w_4 = \frac{8}{35}\psi_9 + \frac{24}{77}\psi_{13}$ $w_5 = -\frac{6}{63}\psi_{11}$ $w_6 = \frac{16}{231}\psi_{13}$

where

$$\psi_0 = f;$$

$$\psi_1 = L^{-1} f;$$

$$\begin{cases} \psi_{2k} = M\psi_{2k-1}, \\ \psi_{2k+1} = L^{-1}\psi_{2k}, \end{cases} \quad k \geq 1.$$

Let us now consider the following linear ordinary integro-differential equation of second order with homogeneous boundary conditions of Dirichlet type

$$\begin{cases} -u''(x) + g(x)u(x) + \varepsilon \int_0^1 K(x,t)u(t)dt = f(x), \\ u(0) = u(1) = 0, \end{cases}$$

where

$$g(x) = \alpha_1(1 + x^2) + \alpha_2(1 + \sin(\pi x)), \quad K(x,t) = e^{x+t}.$$

The above methodology can be realized when the exact solution is

$$u(x) = \alpha_3 \cdot x(1-x) + \alpha_4 \cdot \sin(\pi x)$$

and

$$f(x) = \left(2 \cdot \alpha_3 + \alpha_4 \cdot \pi^2 \sin(\pi x)\right) + (\alpha_1(1 + x^2) + \alpha_2(1 + \sin(\pi x)))$$
$$\times (\alpha_1 \cdot x(1-x) + \alpha_4 \sin(\pi x)) + \varepsilon \left(\alpha_3 \cdot (-e^{x+1} + 3 \cdot e^x) + \alpha_4 \left(\frac{\pi(e+1)e^x}{\pi^2 + 1}\right)\right).$$

We must find approximately L_h^{-1} and M_h operators, h is a mesh size. For this purpose we used second difference quotients and Simpson's complete rule, $mh = 1$.

Table 1

	α_1	0	0	0	1	1	1	1	1
	α_2	0	1	1	0	0	1	1	1
	α_3	1	0	1	0	1	0	1	1
	α_4	0	1	1	1	0	1	0	1
Max	N=10	2.E-07	6.E-03	6.E-03	7.E-03	2.E-07	6.E-03	1.E-07	6.E-03
of	N=100	2.E-11	6.E-05	6.E-05	7.E-05	2.E-11	6.E-05	1.E-11	6.E-05
errors	N=1000	1.E-12	6.E-07	6.E-07	7.E-07	5.E-13	6.E-07	2.E-13	6.E-07

Table 2

	α_1	0	1	0; 1	0	0; 1	1
	α_2	0	0	1	0	0; 1	1
	α_3	0; 1	0; 1	0; 1	1	1	1
	α_4	1	1	1	0	0	0
Max	N=10	8.E-03	7.E-03	6.E-03	2.E-07	2.E-07	1.E-07
of	N=100	8.E-05	7.E-05	6.E-05	2.E-11	2.E-11	1.E-11
errors	N=1000	8.E-07	7.E-07	6.E-07	1.E-12	5.E-13	2.E-13

We emphasize that when $\alpha_2 = \alpha_4 = 0$, the differences between exact and approximate solutions are stipulated by round-off errors.

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