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ON THE APPROXIMATE SOLUTION OF ONE NONLINEAR INTEGRAL EQUATION

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Abstract. The nonlinear integral equation connected with non-linear non-stationary Schrödinger and diffusion equations with the appropriate boundary conditions is considered. The approximate solution of this equation is obtained.

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Introduction. Several Physical processes, such as crystal growth, electron plasmatic waves, nutrient supply in plants and living organisms are described by non-linear Schrödinger type and reaction-diffusion equations with the appropriate boundary conditions [1-9].

Setting of the problem. In $Oxyz$ space we consider the area

$$G = \{-a \leq x \leq a;\ -b \leq y \leq b;\ -c \leq z \leq c\},$$

where $a, b, c > 0$ are the definite constants, and the following equation

$$\Delta U + \lambda U^3 = A_0 U \tag{1}$$

with the boundary condition

$$U|_{\partial G} = 0, \tag{2}$$

where U is unknown function, ∂G is a boundary of G, λ is some parameter $A_0 > 0$ is the definite constant.

Here we will consider the following problem.

Problem 1. In the area G find continuous function U, $(U \neq 0)$, having second order derivatives, satisfying the equation (1) and the condition (2).

Let us rewrite the equation (1) in the form

$$\Delta U - A_0 U = -\lambda U^3. \tag{3}$$

Suppose that the right hand side of the equation (3) is known. According to condition (2) we can use the Poisson formula [10, 11] and equivalently reduce Problem 1 to the following nonlinear integral equation

$$u(x, y, z) = \frac{1}{4\pi} \iiint\limits_G (\lambda u^3) \frac{e^{-mr}}{r} dx' dy' dz', \tag{4}$$

where $m^2 = A_0; r^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$.

Equation (4) is the non-linear homogeneous equation with the weakly singular kernel. We are interested in non-trivial solutions of this equation.

The approximate solution. Now let us consider the following auxiliary problem.

Problem 2. In R^3 to find continuous function U_0, ($U_0 \neq 0$) having second order derivatives, vanishing at infinity and satisfying equation (1).

The approximate solution of Problem 2 is obtained in [12] and is given by the formula

$$U_0 = R \sin e^{-\alpha|x|-\beta|y|-\gamma|z|-D}, \tag{5}$$

where R is the given constant and the constants $\lambda, \alpha, \beta, \gamma > 0$ satisfy the conditions

$$\alpha^2 + \beta^2 + \gamma^2 = A_0, \tag{6}$$

$$\lambda R^2 = \frac{4}{3} A_0, \tag{7}$$

the constant $D > 0$ is chosen for desired accuracy in such a way, that e^{-4D} is negligible (for example for $D = 4,\ e^{-4D} \approx 10^{-7}$).

Note. Let us introduce the notation $\psi = e^{-\alpha|x|-\beta|y|-\gamma|z|-D}$. The first order derivatives of this function has discontinuities at the planes $x = 0; y = 0; z = 0$, but their squares are continuous functions, also the second order derivatives at the eight octants of the space $Oxyz$ exist and the following formulas are valid

$$\Big(\frac{\partial\psi}{\partial x}\Big)^2 = \alpha^2\psi^2; \quad \Big(\frac{\partial\psi}{\partial y}\Big)^2 = \beta^2\psi^2; \quad \Big(\frac{\partial\psi}{\partial z}\Big)^2 = \gamma^2\psi^2;$$

$$\frac{\partial^2\psi}{\partial x^2} = \alpha^2\psi; \quad \frac{\partial^2\psi}{\partial y^2} = \beta^2\psi; \quad \frac{\partial^2\psi}{\partial z^2} = \gamma^2\psi.$$

Taking into the account the formulas

$$\sin\psi \approx \psi - \frac{\psi^3}{6}, \quad \cos\psi \approx 1 - \frac{\psi^2}{2},$$

and putting (5),(6),(7),(8) into (3) we obtain

$$\Big|\Delta U_0 + \lambda U_0^3 - A_0 U_0\Big| = \Big|\Big(1 - \frac{\psi^2}{2}\Big)\Delta\psi - \Big(\psi - \frac{\psi^3}{6}\Big)\Big\{\Big(\frac{\partial\psi}{\partial x}\Big)^2 + \Big(\frac{\partial\psi}{\partial y}\Big)^2 + \Big(\frac{\partial\psi}{\partial z}\Big)^2\Big\}$$

$$+\lambda R^2\psi^3 - A_0\Big(\psi - \frac{\psi^3}{6}\Big)\Big| \leq A_0\psi^5.$$

Hence (5) is the approximate solution of (3) with the accuracy $A_0\psi^5$.

Now if we choose the constants α, β, γ for desired accuracy we obtain the approximate solution of Problem 1 and consequently of equation (4) i.e.

$$\alpha = 16a_0/a, \beta = 16a_0/b, \gamma = 16a_0/c, \tag{8}$$

where

$$a_0^2(1/a^2 + 1/b^2 + 1/c^2) = A_0/256.$$

It is clear from (8) that

$$U_0|_{\partial G} < R10^{-7}.$$

Conclusion. The approximate solutions of equation (4) is given by

$$U_0 = R \sin e^{-\alpha|x|-\beta|y|-\gamma|z|-D},$$

where R is the given constant, the constants $\lambda, \alpha, \beta, \gamma > 0$ satisfy the conditions (6), (7), (8) and $D, \alpha, \beta, \gamma > 0$ are chosen for desired accuracy.

In Fig. 1 and Fig. 2 the profile of U is plotted for the different parameters.

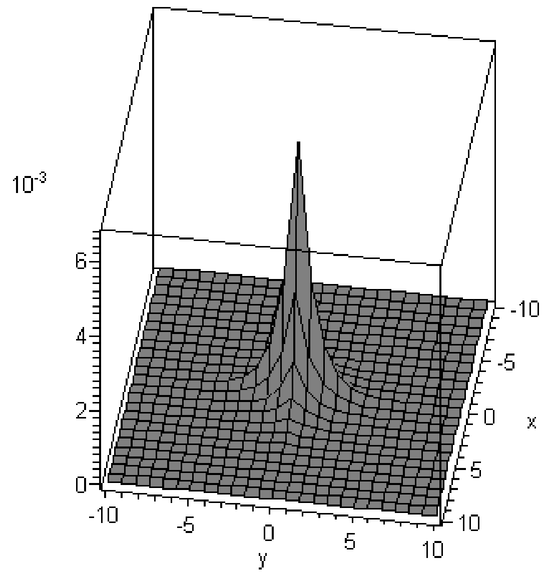

Fig. 1. R=1; z=5; D=4; $\alpha = \beta = \gamma = 1$.

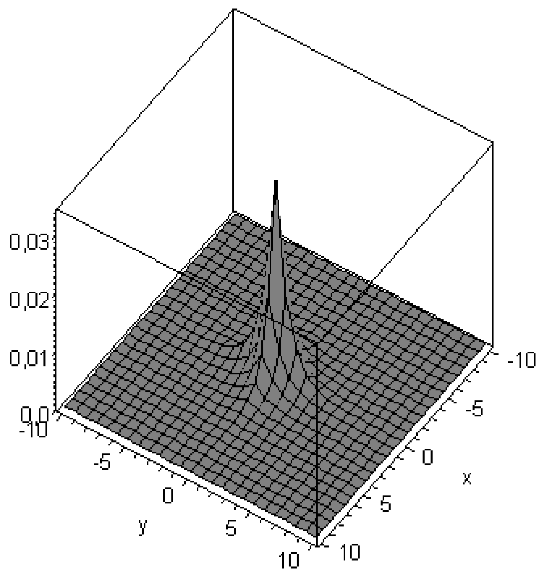

Fig. 2. R=1; z=10; D=4; $\alpha = \beta = \gamma = 1$.

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REFERENCES

1. Alexandrov A., Bogdankevich L., Rukhadze A. Principles of Plasma Electrodynamics. *Springer, Heidelberg, Germany*, 1984.

2. Anderson B.J., Hamilton D.C. Electromagnetic ion cyclotron waves stimulated by modest magnetospheric compressions. *J. Geophys. Res.*, **98**, 11 (1993), 369-11, 382.

3. Gurevich A.V., Shvartsburg A.B. Nonlinear Theory of Radio Wave Propagation in the Ionosphere. *Science, Moscow*, 1973.

4. Lions J.L. Qualques methodes de resolution des problemes aux resolution des problems aux limites non lineares. *Paris*, 1969.

5. Mersmann A. Crystallization Technology Handbook, *CRC*, 2001.

6. Simon B. Schrödinger operators in the twenties cenchery. *J. Math. Phys.*, **41** (2000), 3523-3555.

7. Summers D.R., Thorne M., Xiao F. Relativistic theory of wave-particle resonant diffusion with application to electron acceleration in the magnetosphere. *J. Geophys. Res.*, **103**, 20 (1998), 487-20, 500.

8. Tsintsadze N.L., Kaladze T.D., Van Dam J., Horton W., Fu X.R., Garner T.W. Nonlinear dynamics of the electromagnetic ion cyclotron structures in the inner magnetosphere. *J. of Geophysical Research*, **115**, A07204 (2010), 1-72.

9. Whitham G.B. Linear and Nonlinear Waves. *JOHN WILEY&SONS*, 1974.

10. Tichonov A., Samarsky A. Equations of Mathematical Physics. (Russian) *Moskow, Nauka*, 1966.

11. Bitsadze A. Equations of Mathematical Physics. (Russian) *Moskow, Nauka*, 1970.

12. Khatiashvili N. On the Approximation of the Nonlinear Schrodinger Equation., International J. of Physics and Math. Sciences, CIBTECH www.cibtech.org/jpms.htm, (2012) (to appear).

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