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FINITE ELEMENT METHOD FOR A SYSTEM OF NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS WITH MIXED BOUNDARY CONDITIONS

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Abstract. Galerkin finite element method for the approximation of a system of nonlinear integro-differential equations describing the process of penetrating of a magnetic field into a substance is studied. Initial-boundary value problem with mixed boundary conditions is investigated. The convergence of the finite element scheme is proved. The rate of convergence is given too.

Keywords and phrases: System of nonlinear integro-differential equations, finite elements, Galerkin method.

AMS subject classification: 45K05, 65N30, 35K55.

Mathematical model of the process of penetrating of magnetic field in the substance by Maxwell's system is described [1]. In [2] it was shown that this system can be rewritten in the following form:

$$\frac{\partial W}{\partial t} = -rot\left[a\left(\int\limits_0^t |rotW|^2\,d\tau\right) rotW\right]$$

where $W=(W_1,W_2,W_3)$ is the vector of the magnetic field and the function $a=a(\sigma)$ is defined for $\sigma\in[0,\infty)$.

If the magnetic field has the form $W=(0,u_1,u_2)$ and $u_1=u_1(x,t)$, $u_2=u_2(x,t)$, then we have

$$\begin{aligned}
\frac{\partial u_1}{\partial t} &= \frac{\partial}{\partial x}\left[a\left(\int\limits_0^t\left[\left(\frac{\partial u_1}{\partial x}\right)^2+\left(\frac{\partial u_2}{\partial x}\right)^2\right]d\tau\right)\frac{\partial u_1}{\partial x}\right],\\
\frac{\partial u_1}{\partial t} &= \frac{\partial}{\partial x}\left[a\left(\int\limits_0^t\left[\left(\frac{\partial u_1}{\partial x}\right)^2+\left(\frac{\partial u_2}{\partial x}\right)^2\right]d\tau\right)\frac{\partial u_1}{\partial x}\right].
\end{aligned}\qquad(1)$$

Note that the (1) type model is complex, but special cases of it were investigated [2]-[8]. The existence of global solutions to initial-boundary value problems for such models has been proven in [2]-[5],[8] by using some modifications of the Galerkin method and compactness arguments [9],[10]. For solvability and uniqueness properties of initial-boundary value problems for (1) type models, see [7] as well as many other scientific works.

In [6] some generalization of equations of type (1) is proposed. In this case if the magnetic field again has the form $W=(0,u_1,u_2)$ and $u_1=u_1(x,t)$, $u_2=u_2(x,t)$, then the same process of the magnetic field penetrating into the material is modeled by the following system of integro-differential equations:

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= a\left(\int_0^t\int_0^1\left[\left(\frac{\partial u_1}{\partial x}\right)^2+\left(\frac{\partial u_2}{\partial x}\right)^2\right]dx d\tau\right)\frac{\partial^2 u_1}{\partial x^2}, \\ \frac{\partial u_2}{\partial t} &= a\left(\int_0^t\int_0^1\left[\left(\frac{\partial u_1}{\partial x}\right)^2+\left(\frac{\partial u_2}{\partial x}\right)^2\right]dx d\tau\right)\frac{\partial^2 u_2}{\partial x^2}. \end{aligned} \tag{2}$$

Note that asymptotic behavior of the initial-boundary value problems for (1) and (2) type models were studied in many works (see, for example, [8], [11]-[17]). In [12], [15], [16], [18]-[21] and in a number of other works difference schemes for (1) and (2) type models were investigated. Difference schemes and finite element approximations for a nonlinear parabolic integro-differential scalar model similar to (1) were studied in [21] and [22]. Finite difference schemes and finite element approximations for the scalar equation of (2) type with $a(\sigma) = 1+\sigma$ were studied in [15] and [23], respectively. The convergence of the finite difference approximations of system (2) for the case $a(\sigma) = 1+\sigma$ was studied in [20].

Our aims in the present note are to study the Galerkin finite element approximations of initial-boundary value problem with mixed boundary conditions for system (2).

Consider the following initial-boundary value problem:

$$\frac{\partial u_1}{\partial t} = (1+\sigma)\frac{\partial^2 u_1}{\partial x^2}, \quad \frac{\partial u_2}{\partial t} = (1+\sigma)\frac{\partial^2 u_2}{\partial x^2} \quad (x,t)\in(0,1)\times(0,\infty), \tag{3}$$

$$\begin{aligned} &u_1(0,t) = u_2(0,t) = 0, \quad t\ge 0, \\ &\left.\frac{\partial u_1}{\partial x}\right|_{x=1} = \left.\frac{\partial u_2}{\partial x}\right|_{x=1} = 0, \quad t \ge 0, \end{aligned} \tag{4}$$

$$u_1(x,0) = u_{10}(x), \quad u_2(x,0) = u_{20}(x), \quad x\in[0,1], \tag{5}$$

where

$$\sigma(t) = \int_0^t\int_0^1\left[\left(\frac{\partial u_1}{\partial x}\right)^2+\left(\frac{\partial u_2}{\partial x}\right)^2\right]dx d\tau$$

and $u_{10} = u_{10}(x)$, $u_{20} = u_{20}(x)$ are given functions.

We use the usual norm

$$||u(\cdot,t)||_r = \left\{\int_0^1\sum_{i=0}^r\left|\frac{\partial^i u(x,t)}{\partial x^i}\right|^2 dx\right\}^{1/2}.$$

Now let us construct Galerkin finite element method approximation for the considered problem. One of the ingredients of finite-element method is a variational formulation of the problem. We denote by H the linear space of functions u_1, u_2 satisfying (4) and

$$||u_1(\cdot,t)||_1 < \infty, \quad ||u_2(\cdot,t)||_1 < \infty.$$

The variational formulation of the problem can now be stated as follows: Find a pair of functions $u_1(x,t)$, $u_2(x,t) \in H$ for which

$$< v_1, \frac{\partial u_1}{\partial t} > + < (1+\sigma(t))\frac{\partial u_1}{\partial x}, \frac{\partial v_1}{\partial x} >= \ < f_1, v_1 >,$$
$$< v_2, \frac{\partial u_2}{\partial t} > + < (1+\sigma(t))\frac{\partial u_2}{\partial x}, \frac{\partial v_2}{\partial x} >= \ < f_2, v_2 >, \qquad \forall v_1, v_2 \in H, \tag{6}$$

and

$$< v_1, u_1(x,0) >=< v_1, u_{10}(x) >,$$
$$< v_2, u_2(x,0) >=< v_2, u_{20}(x) >, \quad \forall v_1, v_2 \in H, \tag{7}$$

where

$$< p(x), q(x) >= \int\limits_0^1 p(x)q(x)dx.$$

To approximate the solution of (6) and (7) we require that u_1, u_2 and v_1, v_2 lie in a finite-dimensional subspace S_h of H for each t.

The approximation u_1^h, $u_2^h \in S_h$ to u_1, u_2 is defined by the following variational analog of (6), (7): Find a pair u_1^h, $u_2^h \in S_h$ such that

$$< v_1^h, \frac{\partial u_1^h}{\partial t} > + < (1+\sigma_h(t))\frac{\partial u_1^h}{\partial x}, \frac{\partial v_1^h}{\partial x} >=< f_1, v_1^h >,$$
$$< v_2^h, \frac{\partial u_2^h}{\partial t} > + < (1+\sigma_h(t))\frac{\partial u_2^h}{\partial x}, \frac{\partial v_2^h}{\partial x} >=< f_2, v_2^h >, \quad \forall v_1^h, v_2^h \in S_h, \tag{8}$$

and

$$< v_1^h, u_1^h(x,0) >=< v_1^h, u_{10}(x) >,$$
$$< v_2^h, u_2^h(x,0) >=< v_2^h, u_{20}(x) >, \quad \forall v_1^h, v_2^h \in S_h, \tag{9}$$

where

$$\sigma_h(t) = \int\limits_0^t \int\limits_0^1 \left[\left(\frac{\partial u_1^h}{\partial x}\right)^2 + \left(\frac{\partial u_2^h}{\partial x}\right)^2 \right] dx d\tau.$$

Once a basis has been selected for S_h, (8) and (9) are equivalent to a set of N integro-differential equations, where N is the dimension of S_h.

Theorem. *The error in the finite element approximation u_1^h, u_2^h generated by (8),(9) satisfies the relation*

$$|||u_1 - u_1^h|||_1 + |||u_2 - u_2^h|||_1 \le h^{j-1} \Big\{ c_1 h^2 \left(||u_{10}||^2 + ||u_{20}||^2 \right)$$
$$+ c_2 h^2 \left(\left|\left|\left| \frac{\partial u_1}{\partial t} \right|\right|\right|^2 + \left|\left|\left| \frac{\partial u_2}{\partial t} \right|\right|\right|^2 \right) + c_3 |||u_i|||^2 + c_4 h^{2(j-1)} \sum_{m=1}^{2} |||u_m|||^2$$
$$+ c_5 \left[\sum_{m=1}^{2} \left(c_6 h^{j-1} |||u_m|||^2 + c_7 []u_m[] \right) \right]^2 \Big\}^{1/2},$$

where

$$|||u|||_r = \int\limits_0^T \int\limits_0^1 \sum_{i=0}^{r} \left| \frac{\partial^i E(x,t)}{\partial x^i} \right|^2 dx dt$$

and

$$[]u[] = \int_0^T \int_0^1 |u| dx dt.$$

For the numerical solution of (8),(9) let $\phi_1(x), \ldots, \phi_N(x)$ be a basis for S_h. Therefore u_1^h, $u_2^h \in S_h$ can be represented by

$$u_1^h(x,t) = \sum_{j=1}^N u_{1j}(t)\phi_j(x), \quad u_2^h(x,t) = \sum_{j=1}^N u_{2j}(t)\phi_j(x).$$

Since (8),(9) are valid for all v_1^h, $v_2^h \in S_h$, one can assume $v_1^h = v_2^h = \phi_k$. This yields the following system for the weights $u_1(t)$, $u_2(t)$:

$$Mu_1 + K(u_1,u_2)u_1 = F_1, \quad Mu_2 + K(u_1,u_2)u_2 = F_2, \tag{10}$$

$$Mu_1(0) = U_1, \quad Mu_2(0) = U_2, \tag{11}$$

where

$$M_{jk} =< \phi_k, \phi_j >,$$

$$K(u_1,u_2)_{jk} =< (1+\sigma_h(t))\phi_k', \phi_j' >,$$

$$F_{1j} =< \phi_j, f_1 >, \quad F_{2j} =< \phi_j, f_2 >, \qquad U_{1j} =< \phi_j, u_{10} > \quad U_{2j} =< \phi_j, u_{20} > .$$

To solve the system (10),(11), the algorithm similar to [23] is used. Let us note that in [22] the problem with first kind boundary conditions for (3) type scalar equation is studied. Various numerical experiments were carried out and in all cases the agreement with the theoretical results is observed.

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