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TO NUMERICAL REALIZATIONS OF SOME PROJECTIVE METHODS

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Abstract. A variant of variation-discrete method given in [1, 2] is applied to solve some BVPs with Dirichlet conditions. First the Poisson equation and then the tension-compression problem of a 2D isotropic plate in a square $[-1,1]^2$ is considered. Boundary condition for simplicity are assumed to be homogeneous. It is realized that the method applied has a higher level of accuracy, convergence, stability and a wider class of applicability when compared to the classical finite difference method. Other than those, the scheme obtained consists of four subsystems which can be solved independently and hence enhancing the use of parallel computations.

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The choice of coordinate functions is one of the main problems in approximate methods. For boundary value problems first and a great step was covered by Courant [3]. For approximate solution of BVP of ordinary differential equations the coordinate system presented as a linear combination of classical orthogonal polynomials was used by Mikhlin [4]. There the problems of stability of numerical processes defining as the solution of corresponding algebraic system with respect to coefficients were systematically investigated and calculation of approximate solution was firstly considered. Same problems for multi dimensional BVP for strongly elliptic systems of differential equations in rectangular domains were investigated by Vashakmadze [1, 2].

Let us consider BVP

$$L(\partial_1,\partial_2)u(x,y)=f(x,y),\quad (x,y)\in D:=(-1,1)^2,\quad u|_{\partial D}=0, \tag{1}$$

where $u(x,y)\in C^2(D)\bigcap C(\overline{D})$, $f(x,y)\in C(D)$[1] and $L(\partial_1,\partial_2)$ is a linear strongly elliptic operator. Instead of $u(x,y)$ we take its series expansion

$$u(x,y)=\sum_{i,j=1}^{\infty}u^{ij}\varphi_{ij}(x,y), \tag{2}$$

where u^{ij} are coefficients of $u(x,y)$ in $\varphi_{ij}(x,y)$ bases functions which are defined as multiplication of Legendre polynomials differences (relative to respective indexes) in the following way

$$\varphi_{ij}(x,y):=\chi P_i(x)\chi P_j(y),\quad \chi P_i(x):=\frac{1}{\sqrt{2(2i+1)}}(P_{i+1}(x)-P_{i-1}(x)). \tag{3}$$

[1]For simplicity f is taken from $C(D)$. The only condition f to satisfy is that it is integrable in the general sense over D. Therefore f can be selected from a more general class

The difference is taken in such a way that the homogeneous boundary condition given in (1) is satisfied. It's evident that coordinate functions $\varphi_{ij}(x,y)$ constitute a complete system in an admissible space of classical solutions of (1). The coefficient in operator χ is selected so that after several operations it can be simplified by other coefficients which came out of the integration given in (5). For the numerical realization if we take the first N terms of the series (2), then it becomes

$$\overset{N}{u}(x,y) = \sum_{m,n=1}^{N} u^{mn}\varphi_{mn}. \tag{4}$$

Now if we use Petrov-Galerkin type projective method we have

$$\iint_D L(\partial_1,\partial_2)\overset{N}{u}(x,y)\varphi_{ij}d\omega = \iint_D f(x,y)\varphi_{ij}d\omega =: (f,\varphi_{ij}). \tag{5}$$

(5) gives linear algebraic systems of equations which will be clarified by the following examples. The first example is the Poisson equation and the second one is a 2D tension-compression problem of an isotropic plate.

Example 1. We have the Poisson equation with a unit source function

$$-\Delta u(x,y) = 1, \quad u|_{\partial D} = 0, \quad \overline{D} := [-1,1]^2. \tag{6}$$

Let $I_\triangle := I_{11} + I_{22}$, where $I_{11} := (\partial_{11}\overset{N}{u},\varphi_{ij})$, $\quad I_{22} := (\partial_{22}\overset{N}{u},\varphi_{ij})$.

After inserting (3) into (4) and the resulting equation into the above definition of I_{11} we get

$$I_{11} := \iint_D \left[\frac{\partial^2}{\partial x^2}\left(\sum_{m,n=1}^{N} u^{mn}\chi P_m(x)\chi P_n(y)\right)\chi P_i(x)\chi P_j(y)\right]dxdy. \tag{7}$$

Now, by taking the integral and partial differential operators inside the summation and then integrating by parts while taking into account the following properties of Legendre polynomials

$$\int_{-1}^{1} P_m P_n dt = \frac{2\delta_{mn}}{m+n+1}, \qquad P'_{m+1} - P'_{m-1} = (2m+1)P_m, \tag{8}$$

where prime sign in (8) denotes derivative with respect to the relevant argument x or y, equation (7) reduces to the following algebraic equivalent

$$I_{11} = -u^{i,j}c_j + u^{i,j+2}a_{j+1} + u^{i,j-2}a_{j-1},$$

$$a_j = d_{j+1}\sqrt{d_j d_{j+2}}, \quad c_j = \frac{1}{2}(d_j - d_{j+2}), \quad d_j = \frac{1}{2j-1}.$$

Similarly for the second operator:

$$I_{22} = -u^{i,j}c_i + u^{i+2,j}a_{i+1} + u^{i-2,j}a_{i-1}.$$

After finding algebraic equivalent of $I_\triangle$ the projected approximate equation related to (1) becomes

$$u^{i,j}(c_i + c_j) - u^{i+2,j}a_{i+1} - u^{i-2,j}a_{i-1} - u^{i,j+2}a_{j+1} - u^{i,j-2}a_{j-1} = g^{ij}, \qquad (9)$$

where

$$g^{ij} = \iint\limits_D \chi P_i(x)\chi P_j(y)dxdy, \quad g^{11} = \frac{2}{3}, \quad g^{ij} = 0, \ i \neq 1 \neq j.$$

The system obtained in (9) is in fact consists of four independent subsystems. Indices (i,j) can take either odd or even values from 1 to N. Each combination results in the same type of unknown coefficient indices, hence constitutes an independent subsystem (see Fig. 1a). From the number of members' point of view the obtained scheme resembles the classical finite difference scheme. In the finite difference scheme there are 4 members around a central member $U^{i,j}$ but with a step of one and they cannot form independent subsystems. Using some estimates from ([2], ch. III) it is evident that all eigenvalues of the matrix corresponding to system (9) are positives.

The solution of Poisson BVP is given in Fig. 1 b, c for $N = 3$ and the comparison of the results with Kantorovich and Krilov [5] is shown in Table 1. The results given in terms of $T/(\mu\theta)$. From Table 1 it is seen that even for $N = 2$ the result is equal to the result of Ritz method and for $N = 3$ we have the most accurate result.

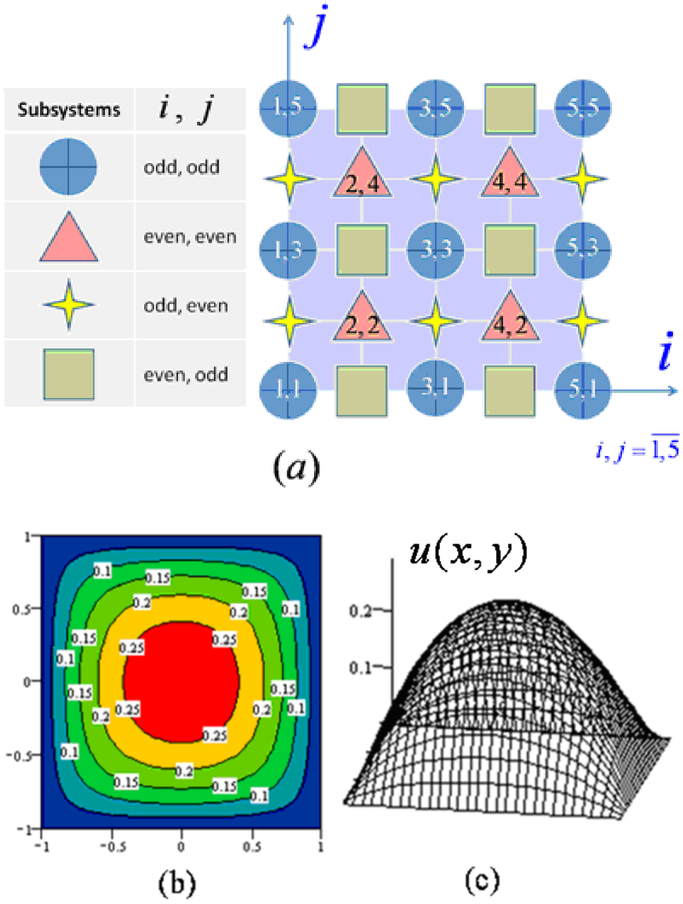

Fig.1. Figurative template for Laplacian(a), solution of the Poisson equation: contour plot(b) and 3-D graph(c).

Table 1. Comparison of the results for the Poisson equation

Methods	Exact	Ritz	Kantorovich and Krilov	Vashakmadze Variation Discrete
$T I(\mu\theta) = 4\iint_D u(x,y)dxdy$	2.244	2.222	2.234	2.222 (for N=2), 2.249 (for N=3)

Example 2. Tension-compression problem of a 2D isotropic plate with homogeneous boundary conditions can be formulated as below (see [1, 2, 6])

$$\mu\Delta\overrightarrow{u} + (\lambda^* + \mu)grad(div\overrightarrow{u}) = \overrightarrow{f}, \quad \overrightarrow{u}|_{\partial D} = 0, \tag{10}$$

where $\overline{D} := [-1,1]^2$, the displacement vector $\overrightarrow{u} = (u_1(x,y), u_2(x,y))^T$, the generalized force function $\overrightarrow{f} = (f_1(x,y), f_2(x,y))^T$, $\lambda^* = 2\lambda\mu(\lambda+2\mu)^{-1}$, λ and μ are Lamé constants.

We already know the template for direct second order derivatives from Example 1, therefore we need only the template for mixed second order derivatives, application of (5) gives

$$\begin{aligned} I_{12} := \iint_D \frac{\partial^2 \overset{N}{u_1}}{\partial x \partial y}\chi P_i(x)\chi P_j(x)dxdy = u_1^{i+1,j+1}b_{i+1,j+1} \\ -u_1^{i+1,j-1}b_{i+1,j} - u_1^{i-1,j+1}b_{i,j+1} + u_1^{i-1,j-1}b_{i,j}, \end{aligned} \tag{11}$$

where $b_{i,j} = \sqrt{d_i d_{i+1} d_j d_{j+1}}$. Considering templates for I_{11}, I_{22} and I_{12} the approximate algebraic equations for (11) become respectively

$$\begin{aligned} -\Big((\lambda^*+2\mu)c_j + \mu c_i\Big)u_1^{i,j} + (\lambda^*+2\mu)\Big(u_1^{i,j+2}a_{j+1} + u_1^{i,j-2}a_{j-1}\Big) \\ +\mu\Big(u_1^{i+2,j}a_{i+1} + u_1^{i-2,j}a_{i-1}\Big) + (\lambda^*+\mu)\Big(u_2^{i+1,j+1}b_{i+1,j+1} \\ -u_2^{i-1,j+1}b_{i,j+1} - u_2^{i+1,j-1}b_{i+1,j} + u_2^{i-1,j-1}b_{i,j}\Big) = g_1^{ij}, \end{aligned} \tag{12}$$

$$\begin{aligned} -\Big((\lambda^*+2\mu)c_i + \mu c_j\Big)u_2^{i,j} + (\lambda^*+2\mu)\Big(u_2^{i+2,j}a_{i+1} + u_2^{i-2,j}a_{i-1}\Big) \\ +\mu\Big(u_2^{i,j+2}a_{j+1} + u_2^{i,j-2}a_{j-1}\Big) + (\lambda^*+\mu)\Big(u_1^{i+1,j+1}b_{i+1,j+1} \\ -u_1^{i-1,j+1}b_{i,j+1} - u_1^{i+1,j-1}b_{i+1,j} + u_1^{i-1,j-1}b_{i,j}\Big) = g_2^{ij}, \end{aligned} \tag{13}$$

where $g_k^{ij} = \iint_D f_k(x,y)\chi P_i(x)\chi P_j(x)dxdy$, $k = 1,2$.

To validate the correctness of the schema obtained in (12) and (13), displacements are taken to be $u_1(x,y) = \chi P_2(x)\chi P_1(y)$ and $u_2(x,y) = u_1(y,x)$. Inserting these test functions into (10) we get the forces as $f_1(x,y) = x\sqrt{15}(-12+15y^2+x^2)/4$, $f_2(x,y) = f_1(y,x)$. After inserting these force functions the algebraic system of equations (12) and (13) are solved and the results are exactly the same as the test functions because they coincide with the coordinate functions, i.e. they are already in the form of multiplication of difference of Legendre polynomials [5].

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