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ON PARABOLIC REGULARIZATION FOR ONE NONLINEAR DIFFUSION MODEL

Gagoshidze M.

Abstract. Parabolic regularization of the one-dimensional analog of Maxwell's system which describes process of penetration of the magnetic field into a substance is studied. Results of numerical experiments based on finite difference schemes are given.

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AMS subject classification: 35Q60, 35K55, 65M06.

Process of penetration of the magnetic field into a substance by the system of Maxwell's equations is described [1]. On the rectangle $[0,1]\times[0,T]$ consider the following initial-boundary value problem for one-dimensional analog of this system:

$$\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\left(a(V)\frac{\partial U}{\partial x}\right), \tag{1}$$

$$\frac{\partial V}{\partial t}=a(V)\left(\frac{\partial U}{\partial x}\right)^2, \tag{2}$$

$$U(0,t)=U(1,t)=0, \tag{3}$$

$$U(x,0)=U_0(x), \quad V(x,0)=V_0(x)\geq Const>0, \tag{4}$$

where a, U_0, V_0 are known functions of their arguments and T is the fixed positive constant. It is well known that system (1),(2) describes many other physical processes (see, for example, [2]).

Let us consider the parabolic regularization of the problem (1)-(4):

$$\frac{\partial U^\varepsilon}{\partial t}=\frac{\partial}{\partial x}\left(a\left(V^\varepsilon\right)\frac{\partial U^\varepsilon}{\partial x}\right), \tag{5}$$

$$\frac{\partial V^\varepsilon}{\partial t}=a\left(V^\varepsilon\right)\left(\frac{\partial U^\varepsilon}{\partial t}\right)^2+\varepsilon\frac{\partial^2 U^\varepsilon}{\partial x^2}, \tag{6}$$

$$U^\varepsilon(0,t)=U^\varepsilon(1,t)=\frac{\partial V^\varepsilon(0,t)}{\partial x}=\frac{\partial V^\varepsilon(1,t)}{\partial x}=0, \tag{7}$$

$$U^\varepsilon(x,0)=U_0(x), \quad V^\varepsilon(x,0)=V_0(x)\geq Const, \tag{8}$$

where ε is the positive constant.

By the first term in the right hand side of equation (6) Joule's rule is described and by the second added term in the same equation thermal conductivity is taken into consideration.

It is well known that parabolic approximations are considered and studied for many mathematical and practical problems (see, for example, [3]).

The system (1),(2), i.e. the model (5),(6) with taking into account only Joule's rule ($\varepsilon = 0$) as well as system (5),(6) with both physical terms ($\varepsilon > 0$) are considered by many authors (see, for example, [4]-[13] and references therein).

Let us note that (5),(6) type parabolic approximation for (1),(2) kind system are considered in [13].

Note that system (1), (2) can be reduced to the integro-differential form [14]. Many works are devoted to the integro-differential models of this type (see, for example, [15]-[19] and references therein). The questions of existence, uniqueness, long time asymptotic behavior of the solutions and numerical resolution of some kind of initial-boundary value problems for integro-differential models of this kind are studied in these works.

The purpose of the present note is to study convergence of the solution $U^\varepsilon, V^\varepsilon$ of problem (5)-(8) to the solution U, V of problem (1)-(4) as $\varepsilon \to 0$. Establishing this convergence by the numerical experiments, using finite difference scheme, constructed and investigated in [12], is also the purpose of this note.

Let us assume that problems (1)-(4) and (5)-(8) have regular solutions. The following statement takes place.

Theorem. *If $a_0 \leqslant a(V) \leqslant A_0$, where a_0, A_0 are positive constants, the solution $U^\varepsilon, V^\varepsilon$ of problem (5)-(8) converges to the solution U, V of problem (1)-(4) as $\varepsilon \to 0$ in the norm of the space $L_2(0,1)$.*

Let us consider problems (1)-(4) and (5)-(8) with the nonhomogeneous right hand sides in both equations of the systems. Introduce the uniform grids $\bar\omega_h = \{t_i = ih, i = 0, 1, ...M\}$ on $[0, 1]$ and $\omega_\tau = \{t_j = j\tau, j = 0, 1, ...N\}$ on $[0, T]$ and using usual notations (see, for example, [20]) let us consider the following difference scheme [12]:

$$u_t^\varepsilon = \left(a\left(\hat v_\varepsilon\right)\hat u_{\bar x}^\varepsilon\right)_x + f, \tag{9}$$

$$v_t^\varepsilon = \sigma a\left(\hat v^\varepsilon\right)\left(\hat u_{\bar x}^\varepsilon\right)^2 + (1-\sigma)\, a\left(v^\varepsilon\right)\left(u_{\bar x}^\varepsilon\right)^2 + \hat v_{\bar x x}^\varepsilon + g, \tag{10}$$

$$u_0^{\varepsilon,j} = u_M^{\varepsilon,j} = v_{x0}^{\varepsilon,j} = v_{\bar x M}^{\varepsilon,j} = 0, \quad j = 0, 1, .., N, \tag{11}$$

$$u_i^{\varepsilon,0} = U_{0,i} \quad v_i^{\varepsilon,0} = V_{0,i}, \quad i = 0, 1, .., M. \tag{12}$$

Here f, g are given functions and $0 \leq \sigma \leq 1$ is a given constant.

The difference scheme (9)-(12) is the first order, i.e. its rate is $O(\tau + h)$.

Note that, in the case $\sigma = 0$, for solving the finite difference scheme (9)-(12) at first we solve system (10) by known tridiagonal matrix algorithm and after we solve system (9) by same algorithm, using in both cases suitable boundary and initial conditions from (11), (12). In the case $\sigma \neq 0$, i.e. for solving fully implicit scheme (9)-(12) we must include a method of solving system of nonlinear algebraic equations. So, it is necessary to use Newton iterative process. It is easy to notice that for the solution of the implicit scheme (9)-(12) the numerical algorithms [4], based on a Newton method [21] without main changes, can be used.

At the end we will note that numerical test experiments, carried out on the basis on difference schemes (9)-(12), are founded on the above described algorithms.

The nonlinearities of the following kinds are considered $a\left(V\right)=\frac{1}{1+V^{\frac{1}{2}}}$. The numerical experiments are quite satisfactory and fully agree with the considered exact test solutions of problem (1)-(4). One of these solutions have the form:

$$U\left(x,t\right)=x\left(1-x\right)\left(1+t\right)\quad V\left(x,t\right)=x^2\left(1-x\right)^2\left(1+t^2\right)+1.$$

The graphs of suitable numerical results are given in Fig. 1 and Fig. 2.

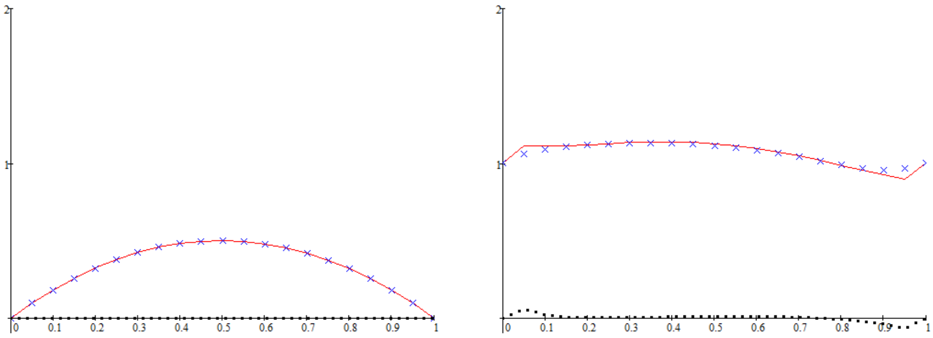

Fig. 1. Exact (solid line) and numerical (marked with $\times$) solutions U (left) and V (right) and differences between exact and numerical solutions (marked with $\bullet$) when $\varepsilon=0.01$.

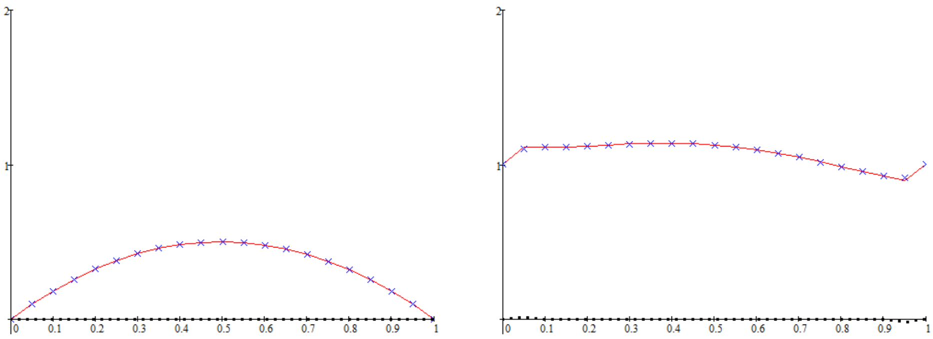

Fig. 2. Exact (solid line) and numerical (marked with $\times$) solutions U (left) and V (right) and differences between exact and numerical solutions (marked with $\bullet$) when $\varepsilon=0.001$.

Let us note that numerical experiments give convergence of the solution u^ε, v^ε of the considered (9)-(12) schemes when $\tau\to 0$, $h\to 0$ and $\varepsilon\to 0$ to the exact solution U, V of the problem (1)-(4).

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Author's address:

M. Gagoshidze
Sokhumi State University
Mathematical Modeling and Computer Sciences
9, Anna Politkovskaya St., Tbilisi 0186
Georgia
E-mail: MishaGagoshidze@gmail.com