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AN ITERATION METHOD FOR A STRING NONLINEAR ORDINARY DIFFERENTIAL EQUATION

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Abstract. A boundary value problem is stated for nonlinear ordinary differential equation, describing the state of a string. The question of convergence and accuracy of one method of the solution of this problem is discussed.

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Introduction. Let us consider the boundary value problem characterizing the static state of a string (see [1,2]):

$$\varphi\left(\int_0^1 (w'(x))^2\,dx\right) w''(x) = f(x), \quad 0 < x < 1, \tag{1}$$

$$w(0) = 0, \quad w(1) = 0. \tag{2}$$

Here $w(x)$ is the unknown displacement function, $f(x)$ and $\varphi(z)$ are the given functions. The first one corresponds to the acting force and the second one is described by stress-strain relations. It is assumed that $\varphi(z)$, $0 \le z < \infty$, is a continuous or differentiable function and satisfy the condition

$$\varphi(z) \ge \alpha > 0, \quad 0 \le z < \infty. \tag{3}$$

When $\varphi(z)$ is a linear function, equation (1) is obtained from Kirchhoff's string oscillation equation by eliminating the time argument. The introduction of the function $\varphi(z)$ enables us not to restrict the consideration to Hook's law in the stress-strain relation (see [3]-[6]).

Method of solution. Assume, that problem (1), (2) have a solution. To find it, we will use M. Chipot's approach (see [3]-[6]). The unknown function is represented as following product

$$w(x) = \lambda v(x), \tag{4}$$

where λ and $v(x)$ are respectively the unknown parameter and the function. The substitution of (4) into (1) gives the following equation

$$\lambda\varphi\left(\int_0^1 ((\lambda v')^2 dx\right) v''(x) = f. \tag{5}$$

Without loss of generality, equation (5) can be replaced by the system of equation

$$v'' = f,$$

$$\lambda\varphi\left(\int_0^1 (\lambda v')^2 dx\right) = 1.$$

If we additionally take into account the (2) boundary conditions and (4) representation, we will obtain the boundary value problem for the function $v(x)$:

$$v'' = f, \tag{6}$$

$$v(0) = 0, \;\; v(1) = 0, \tag{7}$$

which solution has the form

$$v(x) = (x-1)\int_0^x \zeta f(\zeta)d\zeta + x\int_x^1 (\zeta-1)f(\zeta)d\zeta, \tag{8}$$

and nonlinear equation for the parameter λ

$$\lambda\varphi\left(s\lambda^2\right) = 1, \tag{9}$$

where

$$s = \int_0^1 (v')^2 dx. \tag{10}$$

From (8) and (10) it follows, that the value of parameter s is calculated by the formula

$$s = \int_0^1 \left[\int_0^x \zeta f(\zeta)d\zeta + \int_x^1 (\zeta-1)f(\zeta)d\zeta\right]^2 dx. \tag{11}$$

Let us consider the question of existence of the solution of equation (9). By inequality (3) we conclude that its solution is a positive number. Than we transform this equation. After squaring both sides of (9) and then multiplying by s, we get

$$s\lambda^2\varphi^2\left(s\lambda^2\right) = s.$$

Denote $\mu = s\lambda^2$. Then the equation (9) obtains the form

$$\mu\varphi^2(\mu) = s. \tag{12}$$

Let $\varphi(z)$ be a continuous function, $0 \le z < \infty$. Because of $\varphi(z) \ge \alpha > 0$, the root μ of (12) will be from the interval $I = \left[0, \dfrac{s}{\alpha^2}\right]$.

Denote

$$q(\mu) = \mu\varphi^2(\mu) - s, \tag{13}$$

and instead of equation (12) let consider the equation

$$q(\mu) = 0. \tag{14}$$

From (3) and (11), $q(0) \cdot q\left(\frac{s}{\alpha^2}\right) < 0$. Therefore equation (14) has a solution on I. To find it, we will use an approximate algorithm (see [7]).

Approximate algorithm

A1) Let us assume that we can find interval $I_0 \subset I = \left[0, \frac{s}{\alpha^2}\right]$ such, that the equation $q(\mu) = 0$ has a unique root μ^*, $\varphi(\mu) \in C^{(k)}(I)$ and $q'(\mu) \neq 0$, which implies that $\varphi(\mu) \neq 0$, $\varphi(\mu) - 2\mu\varphi'(\mu) \neq 0$. That's why $q(\mu^*) = 0$, we can find the neighborhood of zero I^*, where exists inverse function $g = q^{-1}$, and g have the same smoothness as q.

Denote by N the number of parallel processors. Let us consider the iterative sequence of N-dimensional vectors

$$\left(\mu_1^{(i)},\ \mu_2^{(i)},\ \ldots,\ \mu_N^{(i)}\right), \quad i = 1, 2, \ldots, \tag{15}$$

each component of which is an approximate value of μ^*. Bellow we will show, that when $i \to \infty$ and conditions are defined, all components of vector converges to μ^*.

Let $\left(\mu_1^{(0)},\ \mu_2^{(0)},\ \ldots,\ \mu_N^{(0)}\right)$ - are N initial approximations to μ^* from I_0 and values of function $\varphi(\mu)$ and its first derivatives at these points are given

$$\begin{array}{l} \varphi\left(\mu_1^{(0)}\right) = y_1^{(0)},\ \ldots,\ \varphi\left(\mu_N^{(0)}\right) = y_N^{(0)} \\ \varphi'\left(\mu_1^{(0)}\right),\ \ldots,\ \varphi'\left(\mu_N^{(0)}\right). \end{array}$$

If the values of derivatives of function $y = \varphi(\mu)$ are known, then we can easy find the values of derivatives of inverse function.

Evidently, $\mu^* \equiv g(0)$. For finding the approximations to μ^*, we replace the function $g(y)$ by Hermit's interpolating polynomial, which use as nodes some elements of the set $\left\{y_j^{(i)}\right\}_{j=1}^{N}$, and then compute polynomials value at the point 0.

Denote by M number of nodes used in Hermit's interpolating polynomial, $2 \leq M \leq N-1$. For the j-component of the vector (15) interpolating polynomial of Hermit $H_j^i(y)$ is constructed. We will choose the nodes for $H_j^i(y)$ in the following way: Firstly, let us consider the set of indexes $A = \{1, 2, \ldots, N\}$ and choose N distinct subset A_j of A. The number of elements in each subset is equal to M and $j \in A_j$. For convenience of realization let for all j the Hermit's polynomials have the same order $m - 1$, where $2 \leq m \leq k$.

For approximate solution of equation (12) we consider the following iterative process, where the j-component of the vector (15) is computed in the following way:

$$\mu_j^{(i+1)} = H_j^{(i)}(0), \quad j = \overline{1, N}, \tag{16}$$

where as a nodes of $H_j^i(y)$ are used the points $\left\{y_j^{(i)}\right\}$, $s \in A_j$.

The maximal order of polynomial can be $2M-1$, and minimal - 2.

A2) Let us assume that $\varphi^{(m)}(\mu)$ is bounded on I_0 and $g^{(m)}(y) \neq 0$ on I^*.

Denote by

$$r = \max_{i \leq j \leq N} \left| \mu^* - \mu_j^{(0)} \right|, \quad p = \frac{\left\| g^{(m)}(y) \right\|}{m!} \left\| \varphi'(\mu) \right\|^m .$$

For convergence of (16) the following theorem is true.

Theorem. *Let for function $\varphi(\mu)$ is fulfilled conditions A1), A2), $(k=m)$, and initial values are chosen from area $I_0 \subseteq I$, for points of which $pr^{m-1} < 1$. Let the set A_j contains the points j, $j-1$, $j=\overline{1,N}$. Then the iterative process (16) converges to the solution (12) and the speed of convergence is equal to m (where $(m-1)$ is a polynomials order):*

$$\left| \mu^* - \mu_j^{(i)} \right| \leq \left(\sqrt{p^{\frac{1}{m-1}} r} \right)^{m^i}, \quad j=\overline{1,N}, \quad i \geq i_0.$$

After we compute the approximate value $\bar{\mu}$ of μ^*, we can calculate the correspondence approximate value of parameter λ is the solution of equation (9), according to the formula $\mu = s\lambda^2$.

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