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ON THE GIRSANOV TRANSFORMATION OF BMO MARTINGALES

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Abstract. Using properties of backward stochastic differential equations (BSDEs) we give an alternative proof of the isomorphism of the Girsanov transformation of BMO martingales and improve an estimate of BMO norms.

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Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $F = (\mathcal{F}_t)_{0\le t\le T}$ satisfying the usual conditions of right-continuity and completeness. Throughout the paper M is assumed to be a continuous local P-martingale on the finite time interval $[0, T]$, equal to zero at time $t = 0$.

Recall that (see Kazamaki [2]) a continuous uniformly integrable martingale $(M_t, \mathcal{F}_t)$ with $M_0 = 0$ is from the class BMO if

$$||M||_{BMO} = \sup_{\tau} \Big\| E\big[\langle M\rangle_T - \langle M\rangle_\tau | \mathcal{F}_\tau\big]^{1/2} \Big\|_\infty < \infty,$$

where the supremum is taken over all stopping times $\tau \in [0, T]$ and $\langle M\rangle$ is the square characteristic of M.

Denote by $\mathcal{E}(M)$ the stochastic exponential of a continuous local martingale M

$$\mathcal{E}_t(M) = \exp\{M_t - \frac{1}{2}\langle M\rangle_t\}$$

and let $\mathcal{E}_{\tau,T}(M) = \mathcal{E}_T(M)/\mathcal{E}_\tau(M)$.

Let M be such that $\mathcal{E}(M)$ is a uniformly integrable martingale and define a new probability measure $\tilde{P}$ by $d\tilde{P} = \mathcal{E}_T(M)dP$. To each continuous local P-martingale X we associate the process $\tilde{X} = \langle X, M\rangle - X$, which is a local $\tilde{P}$-martingale according to Girsanov's theorem. We denote this map by $\varphi : \mathcal{L}(P) \to \mathcal{L}(\tilde{P})$, where $\mathcal{L}(P)$ and $\mathcal{L}(\tilde{P})$ are classes of P and $\tilde{P}$ local martingales.

It was proved by Kazamaki [1, 2] that if $M \in BMO(P)$, then $BMO(P)$ and $BMO(\tilde{P})$ are isomorphic under the mapping ϕ and for all $X \in BMO(P)$ the inequality

$$||\tilde{X}||_{BMO(\tilde{P})} \le C_{Kaz}(\tilde{M}) \cdot ||X||_{BMO(P)} \tag{1}$$

is valid (see Theorem 3.6 from [2]), where

$$C^2_{Kaz}(\tilde{M}) = 2p \cdot 2^{1/p} \sup_{\tau} \Big\| E^{\tilde{P}}\Big[\big\{\mathcal{E}_{\tau,T}(\tilde{M})\big\}^{-\frac{1}{p-1}} \Big| \mathcal{F}_\tau\Big] \Big\|_\infty^{(p-1)/p}, \tag{2}$$

$\tilde{M} = \langle M\rangle - M$ and p is such that

$$||\tilde{M}||_{BMO(\tilde{P})} < \sqrt{2}(\sqrt{p}-1). \tag{3}$$

We give an alternative proof of this assertion, which improves also the constant in inequality (1).

Theorem 1. *If $M \in BMO(P)$, then $\phi : X \to \tilde{X}$ is an isomorphism of $BMO(P)$ onto $BMO(\tilde{P})$. In particular, the inequality*

$$\frac{1}{\Big(1+\frac{\sqrt{2}}{2}||M||_{BMO(P)}\Big)}||X||_{BMO(P)} \le ||\tilde{X}||_{BMO(\tilde{P})}$$

$$\le \Big(1+\frac{\sqrt{2}}{2}||\tilde{M}||_{BMO(\tilde{P})}\Big)||X||_{BMO(P)} \tag{4}$$

is valid for any $X \in BMO(P)$.

Proof. Let us consider the process

$$Y_t = E^{\tilde{P}}\big[\langle X\rangle_T - \langle X\rangle_t\big|\mathcal{F}_t\big] = E\big[\mathcal{E}_{t,T}(M)(\langle X\rangle_T - \langle X\rangle_t)\big|\mathcal{F}_t\big]. \tag{5}$$

Since $\langle\tilde{X}\rangle = \langle X\rangle$ under either probability measure, it is evident that

$$||Y||_\infty = ||\tilde{X}||^2_{BMO(\tilde{P})}, \quad \text{where}$$

$$||Y||_\infty = ||Y^*_T||_{L^\infty} \quad \text{and} \quad Y^*_T = \sup_{t\in[0,T]}|Y_t|.$$

For any $X \in BMO(P)$ the process Y is a positive bounded semimartingale with the decomposition

$$Y_t = Y_0 + A_t + \int_0^t \varphi_s dM_s + L_t, \tag{6}$$

where A is a predictable process of bounded variation and L is a local martingale orthogonal to M. It is easy to see that the triple (Y,φ,L) is a solution of the $BSDE$

$$\begin{cases} Y_t = Y_0 - \langle X\rangle_t - \int_0^t \varphi_s d\langle M\rangle_s + \int_0^t \varphi_s dM_s + L_t, \\ Y_T = 0. \end{cases} \tag{7}$$

Let $0 < p < 1$ and $\varepsilon > 0$. Applying the Ito formula for $(Y_\tau+\varepsilon)^p - (Y_T+\varepsilon)^p$ where $0<p<1,\ \varepsilon>0$ and taking conditional expectations we obtain (without loss of generality we assume that all stochastic integrals are martingales, otherwise one can use the localization arguments)

$$\big(Y_\tau+\varepsilon\big)^p - \varepsilon^p = E\Big[\int_\tau^T p(Y_s+\varepsilon)^{p-1}d\langle X\rangle_s\Big|\mathcal{F}_\tau\Big]$$

$$+\frac{p(1-p)}{2}E\Big[\int_\tau^T (Y_s+\varepsilon)^{p-2}d\langle L^c\rangle_s\Big|\mathcal{F}_\tau\Big]$$

$$+E\Big[\int_\tau^T \Big(\frac{p(1-p)}{2}(Y_s+\varepsilon)^{p-2}\varphi_s^2 + p(Y_s+\varepsilon)^{p-1}\varphi_s\Big)d\langle M\rangle_s\Big|\mathcal{F}_\tau\Big]$$
$$-E\Big[\Sigma_{\tau<s\le T}\big((Y_s+\varepsilon)^p-(Y_{s-}+\varepsilon)^p-p(Y_{s-}+\varepsilon)^{p-1}\Delta Y_s\big)\Big|\mathcal{F}_\tau\Big], \tag{8}$$

where, L^c is the continuous martingale part of L.

Because $f(x)=x^p$ is concave for $p\in(0,1)$, the last term in (8) is positive. Therefore, using the inequality

$$\frac{p(1-p)}{2}(Y_s+\varepsilon)^{p-2}\varphi_s^2+p(Y_s+\varepsilon)^{p-1}\varphi_s+\frac{p}{2(1-p)}(Y_s+\varepsilon)^p\ge 0$$

from (8) we obtain

$$(Y_\tau+\varepsilon)^p-\varepsilon^p\ge E\Big[\int_\tau^T p(Y_s+\varepsilon)^{p-1}d\langle X\rangle_s\Big|\mathcal{F}_\tau\Big]$$
$$-\frac{p}{2(1-p)}E\Big[\int_\tau^T (Y_s+\varepsilon)^p d\langle M\rangle_s\Big|\mathcal{F}_\tau\Big]. \tag{9}$$

Since $0<p<1$

$$p\big(||Y||_\infty+\varepsilon\big)^{p-1}E\Big[\langle X\rangle_T-\langle X\rangle_\tau\Big|\mathcal{F}_\tau\Big]\le E\Big[\int_\tau^T p(Y_s+\varepsilon)^{p-1}d\langle X\rangle_s\Big|\mathcal{F}_\tau\Big],$$

from (9) we have

$$p\big(||Y||_\infty+\varepsilon\big)^{p-1}E\Big[\langle X\rangle_T-\langle X\rangle_\tau\Big|\mathcal{F}_\tau\Big]\le\big(Y_\tau+\varepsilon\big)^p-\varepsilon^p+\frac{p}{2(1-p)}E\Big[\int_\tau^T(Y_s+\varepsilon)^p d\langle M\rangle_s\Big|\mathcal{F}_\tau\Big]$$

and taking norms in both sides of the latter inequality we obtain

$$p\big(||Y||_\infty+\varepsilon\big)^{p-1}\cdot||X||^2_{BMO(P)}\le\big(||Y||_\infty+\varepsilon\big)^p-\varepsilon^p+\frac{p}{2(1-p)}\big(||Y||_\infty+\varepsilon\big)^p||M||^2_{BMO(P)}.$$

Taking the limit when $\varepsilon\to 0$ we will have that for all $p\in(0,1)$

$$||X||^2_{BMO(P)}\le\Big(\frac{1}{p}+\frac{1}{2(1-p)}||M||^2_{BMO(P)}\Big)||Y||_\infty.$$

Therefore,

$$||X||^2_{BMO(P)}\le\min_{p\in(0,1)}\Big(\frac{1}{p}+\frac{1}{2(1-p)}||M||^2_{BMO(P)}\Big)||Y||_\infty$$
$$=\Big(1+\frac{\sqrt{2}}{2}||M||_{BMO(P)}\Big)^2||Y||_\infty, \tag{10}$$

since the minimum of the function $f(p)=\frac{1}{p}+\frac{1}{2(1-p)}||M||^2_{BMO(P)}$ is attained for $p^*=\sqrt{2}/(\sqrt{2}+||M||_{BMO(P)})$ and $f(p^*)=\Big(1+\frac{\sqrt{2}}{2}||M||_{BMO(P)}\Big)^2$.

Thus, from (10)

$$\frac{1}{\left(1+\frac{\sqrt{2}}{2}||M||_{BMO(P)}\right)}||X||_{BMO(P)} \le ||\tilde{X}||_{BMO(\tilde{P})}.$$

Now we can use inequality (10) for the Girsanov transform of $\tilde{X}$.

Since $dP/d\tilde{P} = \mathcal{E}_T^{-1}(M) = \mathcal{E}_T(\tilde{M})dP$, $\tilde{M}, \tilde{X} \in BMO(\tilde{P})$ and

$$\varphi(\tilde{X}) = \langle \tilde{X}, \tilde{M} \rangle - \tilde{X} = X,$$

from (10) we get the inverse inequality:

$$||\tilde{X}||_{BMO(\tilde{P})} \le \left(1+\frac{\sqrt{2}}{2}||\tilde{M}||_{BMO(\tilde{P})}\right)||X||_{BMO(P)}. \quad \square$$

Let us compare the constant

$$C(\tilde{M}) = 1+\frac{\sqrt{2}}{2}||\tilde{M}||_{BMO(\tilde{P})}$$

from (4) with the corresponding constant $C_{Kaz}(\tilde{M})$ from (1) (Kazamaki [2]).

Since $2^{1/p} > 1$ and by Jensen's inequality

$$E^{\tilde{P}}\left[\left\{\mathcal{E}_{\tau,T}(\tilde{M})\right\}^{-\frac{1}{p-1}}\Big|\mathcal{F}_\tau\right] \ge 1,$$

the constant $C_{Kaz}(\tilde{M})$ is more than $\sqrt{2p}$, where from (3) $p > \left(1+\frac{\sqrt{2}}{2}||\tilde{M}||_{BMO(\tilde{P})}\right)^2$. Therefore, we obtain that at least

$$C^2(\tilde{M}) < \frac{1}{2}C^2_{Kaz}(\tilde{M}).$$

REFERENCES

1. Kazamaki N. On transforming the class of BMO-martingales by a change of law. *Tohoku Mathematical Journal*, **31** (1979), 117-125.

2. Kazamaki N. Continuous Exponential Martingales and BMO. **1579** *Lecture Notes in Mathematics, Springer, Berlin-Heidelberg*, 1994.

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