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# ASYMPTOTIC BEHAVIOR OF THE SOLUTION AND FINITE DIFFERENCE SCHEME FOR ONE NONLINEAR INTEGRO-DIFFERENTIAL MODEL WITH SOURCE TERMS

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**Abstract**. One nonlinear integro-differential system with source terms is considered. The model arises at describing penetration of a magnetic field into a substance and is based on the well known Maxwell system. Large time behavior of solutions of the initial-boundary value problem is studied. Corresponding finite difference scheme is considered as well.

**Keywords and phrases**: Nonlinear integro-differential equation, asymptotic behavior, finite difference scheme.

**AMS subject classification**: 45K05, 65M06, 35K55.

One kind of nonlinear integro-differential model arises on mathematical simulation of the process of penetration of a magnetic field into a substance [1]. This model was introduced after reduction of well known nonlinear Maxwell's differential system [2] to the integro-differential form. In [3] some generalization of such type models is given. One-dimensional simple analog called by averaged integro-differential model by the author describing the same physical process has the following form

$$\frac{\partial U}{\partial t} - a\left(\int_0^t \int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx d\tau\right)\frac{\partial^2 U}{\partial x^2} = 0, \qquad (1)$$

where $a = a(S) \geq a_0 = Const > 0$ is a given function of its argument.

Many works are dedicated to the investigation and numerical resolution of the integro-differential models described in [1] and [3]. Especially, in [1], [3]-[10] solvability and uniqueness of the initial-boundary value problems for equations and systems of this type are studied. Asymptotic behavior of solutions as $t \to \infty$ is investigated in many works as well (see, for example, [8],[10]-[25] and references therein). Numerical resolution by finite difference scheme is given in [10], [14], [17]-[24], [26], [27] and in a number of other papers as well.

The aim of this note is to study asymptotic behavior of solution $t \to \infty$ and to construct approximate solutions for one generalization of the system of type (1) by adding monotonic nonlinear terms. This system has the form:

$$\begin{aligned}
&\frac{\partial U}{\partial t} - \left(1 + \int_0^t \int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] dx d\tau\right)\frac{\partial^2 U}{\partial x^2} + |U|^{q-2}U = 0,\\
&\frac{\partial V}{\partial t} - \left(1 + \int_0^t \int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] dx d\tau\right)\frac{\partial^2 V}{\partial x^2} + |V|^{q-2}V = 0,
\end{aligned} \qquad (2)$$

where $q \geq 2$.

Let us note that generalizations of such kind for the equation described in [1] are made in [22] and for (2) type averaged equation is discussed in [24].

In $[0,1] \times [0, \infty)$ let us consider the following initial-boundary value problem

$$\begin{gathered} U(0,t) = U(1,t) = V(0,t) = V(1,t) = 0, \\ U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \end{gathered} \tag{3}$$

where $U_0 = U_0(x)$ and $V_0 = V_0(x)$ are given functions.

The following statement is true.

**Theorem 1.** *If $q \geq 2$ and $U_0, V_0 \in H_0^1(0,1)$, then problem (2),(3) does not have more than one solution and the following asymptotic property takes place*

$$\|U\| + \left\| \frac{\partial U}{\partial x} \right\| + \|V\| + \left\| \frac{\partial V}{\partial x} \right\| \leq C \exp\left(-\frac{t}{2}\right).$$

Here $\|\cdot\|$ is the usual norm of the space $L_2(0,1)$ and $C$ denotes the positive constant, independent of $t$.

On $[0,1] \times [0,T]$ let us introduce a grid with mesh points denoted by $(x_i, t_j) = (ih, j\tau)$, where $i = 0, 1, ..., M$; $j = 0, 1, ..., N$, with $h = 1/M$, $\tau = T/N$. The initial line is denoted by $j = 0$. The discrete approximation at $(x_i, t_j)$ is designed by $u_i^j$, $v_i^j$ and the exact solution to problem (2), (3) by $U_i^j$, $V_i^j$. We will use the following known notations:

$$r_{t,i}^j = \frac{r_i^{j+1} - r_i^j}{\tau}, \quad r_{\bar{t},i}^j = r_{t,i}^{j-1} = \frac{r_i^j - r_i^{j-1}}{\tau}.$$

Using usual methods of construction of difference schemes (see, for example, [28]) let us construct, as in [17], [21] for the same problem without the force terms, the following finite difference scheme for problem (2),(3):

$$\begin{gathered} \frac{u_i^{j+1} - u_i^j}{\tau} - \left(1 + \tau h \sum_{k=1}^{j+1} \sum_{l=1}^{M} \left[ \left(u_{\bar{x},l}^k\right)^2 + \left(v_{\bar{x},l}^k\right)^2 \right] \right) u_{\bar{x}x,i}^{j+1} + |u_i^{j+1}|^{q-2} u_i^{j+1} = 0, \\ \frac{v_i^{j+1} - v_i^j}{\tau} - \left(1 + \tau h \sum_{k=1}^{j+1} \sum_{l=1}^{M} \left[ \left(u_{\bar{x},l}^k\right)^2 + \left(v_{\bar{x},l}^k\right)^2 \right] \right) v_{\bar{x}x,i}^{j+1} + |v_i^{j+1}|^{q-2} v_i^{j+1} = 0, \\ i = 1, 2, ...M-1; \quad j = 0, 1...N-1, \\ u_0^j = u_M^j = v_0^j = v_M^j = 0, \;\; j = 0, 1, \ldots, N, \\ u_i^0 = U_{0,i}, \; v_i^0 = V_{0,i}, \;\; i = 0, 1, \ldots, M. \end{gathered} \tag{4}$$

The following statement takes place.

**Theorem 2.** *If $q \geq 2$ and the initial-boundary value problem (2),(3) has the sufficiently smooth solution $U = U(x,t)$, $V = V(x,t)$, then the finite difference scheme (4) converges and the following estimate is true*

$$\left\| u^j - U^j \right\| + \left\| v^j - V^j \right\| \leq C(\tau + h).$$

Here $\|\cdot\|$ is a discrete analog of the norm of the space $L_2(0,1)$ and $C$ is a positive constant, independent of $\tau$ and $h$.

Note that for solving the finite difference scheme (4) we use an algorithm analogical to [17], [21]. So, it is necessary to use Newton iterative process [29]. According to this method the great numbers of numerical experiments are carried out. These experiments agree with the theoretical results given in Theorems 1 and 2.

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