Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 25, 2011

BOUNDARY VALUE PROBLEMS OF STATICS IN THE THEORY OF THERMOELASTICITY WITH MICROTEMPERATURES FOR AN ELASTIC PLANE WITH A CIRCULAR HOLE

Tsagareli I., Svanadze M.

Abstract. In the present work we solve explicitly, by means of absolutely and uniformly convergent series the boundary value problems of statics of the linear theory of thermoelasticity with microtemperatures for an elastic plane with a circular hole. The question on the uniqueness of a solution of the problem is investigated.

Keywords and phrases: Thermoelasticity, microtemperatura, boundary value problem, explicit solution.

AMS subject classification: 74F05, 74G30, 74G10.

1. Basic equations. We consider the plane D with a circular hole. Let R be the radius of the boundary S. The system of equations of the theory of thermoelasticity with microtemperatures is the form [1,2]:

$$\begin{aligned}
&\mu\Delta u(x)+(\lambda+\mu)graddivu(x)=\beta gradu_3(x),\\
&k\Delta u_3(x)+k_1divw(x)=0,\\
&k_6\Delta w(x)+(k_4+k_5)graddivw(x)-k_3gradu_3(x)-k_2w(x)=0,
\end{aligned}\tag{1}$$

where $\lambda,\mu,\beta,k,k_1,k_2,k_3,k_4,k_5,k_6$ are constitutive coefficients [1]; $u(x)$ is the displacement of the point x; $u=(u_1,u_2); w=(w_1,w_2)$ is the microtemperatures vector; u_3 is temperature measured from the constant absolute temperature T_0; Δ is the Laplace operator.

Problems. Find a regular solution $U(u,u_3,w)$ of system (1) satisfying the boundary conditions

$$\begin{aligned}
&I.u(z)=f(z),\quad u_3(z)=f_3(z),\quad w(z)=p(z);\\
&II.T'(\partial_z,n)u(z)-\beta u_3(z)n(z)=f(z),\quad k\partial_nu_3(z)+k_1w(z)n(z)=f_3(z),
\end{aligned}\tag{2}$$

$$T''(\partial_z,n)w(z)=p(z),\quad z\in S,$$

where n is the external unit normal vector to S; $f=(f_1,f_2),p=(p_1,p_2),f_1,f_2,f_3,p_1,p_2$ - are the given functions on S, $\partial_n=\frac{\partial}{\partial_n}$; $\partial_k=\frac{\partial}{\partial_{x_k}}$, $k=1,2$; $T'u$ is the stress vector in the classical theory of elasticity; $T''w$ is stress vector for microtemperatures [2]:

$$T'(\partial_x,n)u(x)=\mu\partial_nu(x)+\lambda n(x)divu(x)+\mu\sum_{i=1}^{2}n_i(x)gradu_i(x),$$

$$T''(\partial_x,n)w(x)=(k_5+k_6)\partial_nw(x)+k_4n(x)divw(x)+k_5\sum_{i=1}^{2}n_i(x)gradw_i(x).$$

Vector $U(x)$ satisfies the following conditions at infinite:

$$(u(x),w(x))=O(1),\ \ r^2\partial_{x_k}(u(x),w(x))=O(1),\ \ r^2u_3(x)=O(1),\ \ k=1,2, \tag{3}$$

where $r^2=x_1^2+x_2^2$. Separately we will study the following problems: 1. Find in a plane D solution $u(x)$ of equation $(1)_1$, if on the circumference S there are given the values: a) of the vector u - problem A_1; b) of the vector $T'(\partial_z,n)u(z)-\beta u_3(z)n(z)$ - problem A_2.
2. Find in a plane D solutions $u_3(x)$ and $w(x)$ of the system of equations $(1)_2$ and $(1)_3$, if on the circumference S there are given the values: a) of the function u_3 and the vector $w(z)$ - problem B_1; b)of the function $k\partial_n u_3(z)+k_1w(z)n(z)$ and the vector $T''(\partial_z,n)w(z)$ - problem B_2.

Thus the above-formulated problems of thermoelasticity with microtemperatures can be considered as a union of two problems: I - (A_1,B_1) and II - (A_2,B_2).

2. Uniqueness theorems. By virtue of conditions (3), the following theorems are valid.

Theorem 1. *The difference of two arbitrary solutions of problem I is equal to zero:* $u_1=u_2=0,\quad u_3=0,\quad w_1=w_2=0;$

Theorem 2. *The difference of two arbitrary solutions of problem II is the vector $U(u_1(x),u_2(x),u_3(x),w_1(x),w_2(x))$, where $u_1=q_1,\quad u_2=q_2,\quad u_3=c,\quad w_1=w_2=0,\quad c,q_1,q_2$ is an arbitrary constants.*

3. Solutions of the Problems. On the basic of the system $(1)_2,(1)_3$, we can write

$$\triangle(\triangle+s_1^2)u_3=0,\quad \triangle(\triangle+s_1^2)divw=0.$$

Solutions of these equations are represented in the form [3]:

$$u_3(x)=\varphi_1(x)+\varphi_2(x),\quad w_1(x)=a_1\partial_1\varphi_1(x)+a_2\partial_2\varphi_2(x)-a_3\partial_2\varphi_3(x),$$

$$w_2(x)=a_1\partial_2\varphi_1(x)+a_2\partial_1\varphi_2(x)+a_3\partial_1\varphi_3, \tag{4}$$

where $\triangle\varphi_1=0,(\triangle+s_1^2)\varphi_2=0,(\triangle+s_2^2)\varphi_3=0,s_1^2=-\dfrac{kk_2-k_1k_3}{kk_7},s_2^2=-\dfrac{k_2}{k_6}$, $a_1=-\dfrac{k_3}{k_2},a_2=-\dfrac{k}{k_1},a_3=\dfrac{k_6}{k_7};\quad k_7=k_4+k_5+k_6;\quad k,k_2,k_6,k_7>0$ [2].

Problem B_1. Taking into account formulas (4), the boundary conditions of the problem B_1 can be rewritten as:

$$u_3(z)=f_3(z),\quad w_n(z)=p_n(z),\quad w_s(z)=p_s(z), \tag{5}$$

where $w_n=(w\cdot n),w_s=(w\cdot s),p_n=(p\cdot n),p_s=(p\cdot s),n=(n_1,n_2),s=(-n_2,n_1)$.

The harmonic function φ_1 and metaharmonic functions φ_2 and φ_3 are represented in the form of series:

$$\varphi_1(x)=\frac{1}{2}Y_{01}+\sum_{m=1}^{\infty}\left(\frac{R}{r}\right)^m(Y_{m1}\cdot\nu_m(\psi)),$$

$$\varphi_2(x)=\sum_{m=0}^{\infty}K_m(s_2r)(Y_{m2}\cdot\nu_m(\psi)),\quad \varphi_3(x)=\sum_{m=0}^{\infty}K_m(s_3r)(Y_{m3}\cdot\ s_m(\psi)),\tag{6}$$

respectively, where K_m is the Bessel's function with an imaginary argument; Y_{mk} are the unknown two-component constants vectors, $\nu_m(\psi)=(\cos m\psi,\sin m\psi)$, $s_m(\psi)=(-\sin m\psi,\cos m\psi)$, $\quad k=1,2,\quad m=0,1,....$

Let the functions p_n,p_s and f_3 expand into the Fourier series.

We substitute (6) into (4) and then the obtained expression into (5). Passing to the limit, as $r\to R$, for the unknowns Y_{mk} we obtain a system of algebraic equations:

$$\begin{aligned}
&-ma_1Y_{m1}+Ra_2s_2K_m'(s_2R)Y_{m2}+a_3mK_m(s_3R)Y_{m3}=\alpha_mR,\\
&ma_1Y_{m1}+a_2mK_m(s_2R)Y_{m2}+Ra_3s_3K_m'Y_{m3}=\beta_mR,\\
&Y_{m1}+K_m(s_2R)Y_{m2}=\gamma_m,\quad m=1,2,...;
\end{aligned}\tag{7}$$

$$Y_{01}=\gamma_0-\frac{\alpha_0K_0(s_2R)}{a_2s_2K_0'(s_2R)},\quad Y_{02}=\frac{\alpha_0}{2a_2s_2K_0'(s_2R)},\quad Y_{03}=\frac{\beta_0}{2a_3s_3K_0'(s_3R)},\tag{8}$$

where $\alpha_m=(\alpha_{m1},\alpha_{m2})$, $\quad\beta_m=(\beta_{m1},\beta_{m2})$ and $\gamma_m=(\gamma_{m1},\gamma_{m2})$ ate the Fourier coefficients of the functions p_n,p_s and f_3, respectively.

Relying on the theorem on the uniqueness of a solution of the problem we can conclude that the principal determinants of the system (7) are other than zero. Substituting the solution of the systems (7) and solution (8) into (6) and then into (4),we can find values of the functions $u_3(x),w_1(x)$ and $w_2(x)$.

Problem B_2. Taking into account formulas (4), the boundary conditions of the problem B_2 can be rewritten as:

$$\begin{aligned}
&k_7\left[\partial_rw_n\right]_R+\tfrac{k_4}{R}\left[\partial_\psi w_s\right]_R=p_n(z),\quad k_6\left[\partial_rw_s\right]_R+\tfrac{k_5}{R}\left[\partial_\psi w_n\right]_R=p_s(z),\\
&k\left[\partial u_3\right]_R+k_1[w_n]_R=f_3(z).
\end{aligned}\tag{9}$$

We substitute (6) into (4), then the obtained expression into (9). Passing to the limit, as $r\to R$, from (9) we obtain the system of linear algebraic equations with regard to the unknowns Y_{mk} for every value m:

$$a_1m[(k_7-k_4)m+k_7]Y_{m1}+a_2[k_7s_2^2K_m''(s_2R)R^2-k_4m^2a_1K_m(s_2R)]Y_{m2}$$

$$+a_3m[k_7[s_3RK_m'(s_3R)-K_m(s_3R)]-k_4s_3RK_m'(s_3R)]Y_{m3}=\alpha_mR^2,$$

$$a_1m[(k_6+k_5)m-k_6]Y_{m1}+a_2m[k_6[s_2RK_m'(s_2R)-K_m(s_2R)]+k_5s_2RK_m'(s_2R)]Y_{m2}$$

$$+a_3[k_6s_3^2R^2K_m''(s_3R)+k_5a_3m^2K_m(s_3R)]Y_{m3}=\beta_mR^2,$$

$$m(-k+k_1a_1)Y_{m1}+s_2K_m'(s_2R)(k+k_1a_2R)Y_{m2}+a_3mK_m(s_3R)Y_{m3}=\gamma_mR,m=1,2,....$$

Taking into account the condition $\int_S p(y)d_yS=0$ and equation $(1)_2$, we obtain: $Y_{02}=0,Y_{03}=0,Y_{01}=const.$

Problem A_1. A solution $(1)_1$ is sought in the form

$$u(x)=v_0(x)+v(x),\tag{10}$$

where v_0 is a particular solution of equation $(1)_1$, and v is a general solution of the corresponding homogeneous equation $(1)_1$. Direct checking shows that v_0 has the form: $v_0(x) = \frac{\beta}{\lambda+2\mu} grad[-\frac{1}{s_1^2}\varphi_2(x) + \varphi_0(x)]$, where φ_0 is a biharmonic function: $\triangle\varphi_0 = \varphi_1$.

A solution $v(x) = (v_1(x), v_2(x))$ of the homogeneous equation corresponding to $(1)_1$: $\mu\triangle v(x) + (\lambda+\mu) graddivv(x) = 0$ is sought in the form

$$v_1(x) = \partial_1[\Phi_1(x) + \Phi_2(x)] - \partial_2\Phi_3(x), \quad v_2(x) = \partial_2[\Phi_1(x) + \Phi_2(x)] + \partial_1\Phi_3(x), \qquad (11)$$

where $\Delta\Phi_1(x) = 0, \Delta\Delta\Phi_2(x) = 0, \Delta\Delta\Phi_3(x) = 0, (\lambda+2\mu)\partial_1\Delta\Phi_2(x) - \mu\partial_2\Delta\Phi_3(x) = 0, (\lambda+2\mu)\partial_2\Delta\Phi_2(x) + \mu\partial_1\Delta\Phi_3(x) = 0$, Φ_1, Φ_2, Φ_3 are the scalar functions.

We can represent the harmonic function Φ_1 and biharmonic functions Φ_2 and Φ_3 in the form

$$\begin{aligned} &\Phi_1(x) = \sum_{m=0}^{\infty} \left(\frac{R}{r}\right)^m (X_{m1} \cdot \nu_m(\psi)), \quad \Phi_2(x) = \sum_{m=0}^{\infty} R^2 \left(\frac{R}{r}\right)^{m-2} (X_{m2} \cdot \nu_m(\psi)), \\ &\Phi_3(x) = \frac{R^2(\lambda+2\mu)}{\mu} \sum_{m=0}^{\infty} \left(\frac{R}{r}\right)^{m-2} (X_{m2} \cdot s_m(\psi)), \end{aligned} \qquad (12)$$

where X_{mk} are the unknown two-component vectors, $k = 1, 2$.

Taking into account (10) the condition $(2)_I$, we can write as: $v(z) = \Psi(z)$, where $\Psi(z) = f(z) - v_0(z)$ is the known vector.

Substitute in this boundary condition the formulas (11) and (11), we obtain the system of algebraic equations for every m, whose solution is written as follows:

$$X_{01} = \frac{\eta_0 R}{4}, X_{02} = \frac{\varsigma_0 R}{4(\lambda+2\mu)}, X_{m1} = \frac{\eta_m R}{m} - \frac{(\varsigma_m - \eta_m)R}{2(\lambda+\mu)m}, X_{m2} = \mu\frac{(\varsigma_m - \eta_m)R}{2(\lambda+\mu)m},$$

where η_m and ς_m are the Fourier coefficients of the functions $\Psi_n(z)$ and $\Psi_s(z)$; Ψ_n and Ψ_s are normal and tangential components of the function $\Psi(z)$, respectively.

Problem A_2. Taking into account (10) the condition $(2)_{II}$, we can rewrite as $T'(\partial_z, n)v(z) = \Psi(z)$, where $\Psi(z) = f(z) + \beta u_3(z)n(z) - T'(\partial_z, n)v_0(z)$ is the known vector, $\Psi = (\Psi_1, \Psi_2)$.

We substitute in this boundary condition the formulas (11) and (12). For the unknowns X_{m1} and X_{m2} we obtain a system of algebraic equations whose solution has the form

$$X_{01} = \frac{\eta_0 R^2}{4(\lambda+2\mu)}, \quad X_{02} = \frac{\varsigma_0 R^2}{4(\lambda+2\mu)}, \quad X_{m1} = \frac{R^2}{c_3}\varsigma_m - \frac{c_4 R^2}{c_2 c_3 - c_1 c_4}(\mu\eta_m - c_1\varsigma_m),$$

$$X_{m2} = \frac{c_4 R^2}{c_2 c_3 - c_1 c_4}(\mu\eta_m - c_1\varsigma_m),$$

where $c_1 = \mu[2(\lambda+\mu)m^2 - (\lambda+2\mu)m]$, $\quad c_2 = 2(\lambda+\mu)(\lambda+3\mu)m^2 + (\lambda+2\mu)[(3\lambda+5\mu)m + 2\mu]$, $\quad c_3 = m\mu(2\mu-1)$, $\quad c_4 = 2(\lambda+3\mu)m(2m+3) + 2(\lambda+2\mu), m = 1, 2, \ldots$. η_m and ς_m are the Fourier coefficients of respectively normal and tangential components of the function $\Psi(z)$.

Having solved problems A_1, A_2, B_1 and B_2, we can write solutions of the initial problems I and II.

Acknowledgement. The designated project has been fulfilled by financial support of the Shota Rustaveli National Science Foundation (Grant GNSF/ST 08/3-388). Any idea in this publication is possessed by the author and may not represent the opinion of Shota Rustaveli National Science Foundation itself.

REFERENCES

1. Iesan D., Quintanilla R. On the theory of thermoelasticity with microtemperatures. *J. Thermal Stresses*, **23** (2000), 199-215.

2. Scalia A., Svanadze A., Tracina R. Basic theorems in the equilibrium theory of thernoelasticity with microtemperatures, *J.Thermal Stresses*, **33** (2010), 721-753.

3. Vekua I. On metaharmonic functions. *Tr. Tbilis. Mat. Inst.*, **12** (1943), 105-174.

Received 3.06.2011; revised 17.09.2011; accepted 30.10.2011.

Authors' addresses:

Tsagareli I.
I. Vekua Institute of Applied Mathematics of
Iv. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia
E-mail: i.tsagareli@yahoo.com

M. Svanadze
Institute of Mathematics, University of Gottingen
3-5, Bunsenstrasse, Gottingen, D-37078
Germany
E-mail: maia.svamadze@gmail.com