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EXPLICIT SOLUTIONS OF SOME PROBLEMS OF STATICS OF THE LINEAR THEORY OF ELASTIC MIXTURES FOR HALF-PLANE

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Abstract. Using the method of Kolosov-Muskhelishvili for a half-plane are explicity solved the following two boundary value problem of statics of the linear theory of elastic mixtures, on the case, when on boundary domain are given $(U_n,\sigma_s)^T$ and $(U_s,\sigma_n)^T$ vectors respectively, where $U_n=(u_1n_1+u_2n_2,u_3n_1+u_4n_2)^T$, $U_s=(u_2n_1-u_1n_2,u_1n_1-u_3n_2)^T$, $\sigma_n=((Tu)_2n_1+(Tu)_1n_2,(Tu)_3n_1+(Tu)_4n_2)^T$, $\sigma_s=((Tu)_2n_1-(Tu)_1n_2,(Tu)_4n_1-(Tu)_3n_2)^T$, $U_k(Tu)_k$, $k=\overline{1,4}$, are displacement and stress vectors components respectively, $n=(n_1,n_2)^T$ is the unit vector of the outer normal.

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1. A homogeneous equation of statics of the theory of elastic mixtures in a complex form is of the type [1]

$$\frac{\partial^2 U}{\partial z\partial \overline{z}}+\mathcal{K}\frac{\partial^2 \overline{U}}{\partial \overline{z}^2}=0, \tag{1}$$

where $z=x_1+ix_2$, $\overline{z}=x_1-ix_2$, $\dfrac{\partial}{\partial z}=\dfrac{1}{2}\Big(\dfrac{\partial}{\partial x_1}-i\dfrac{\partial}{\partial x_2}\Big)$, $\dfrac{\partial}{\partial \overline{z}}=\dfrac{1}{2}\Big(\dfrac{\partial}{\partial x_1}+i\dfrac{\partial}{\partial x_2}\Big)$, $U=(u_1+iu_2,u_3+iu_4)^T$, $u'=(u_1,u_2)^T$ and $u''=(u_3,u_4)^T$ are partial displacements,

$$\mathcal{K}=-\frac{1}{2}em^{-1},\quad e=\begin{bmatrix}e_4 & e_5\\ e_5 & e_6\end{bmatrix},\quad m^{-1}=\begin{bmatrix}m_1 & m_2\\ m_2 & m_3\end{bmatrix}^{-1},$$

m_k, l_{3+k}, $k=1,2,3$, are expressed in the therms of the elastic [1].

In the case of an infinite domain in addition to the conditions of regularity it is necessary to impose the requi rements at infinity

$$U=O(1),\quad \frac{\partial u}{\partial x_k}=O(|x|^{-2}),\quad k=1,2,\quad |x|^2=x_1^2+x_2^2.$$

In [1] M. Basheleishvili obtained the representations:

$$U=(u_1+iu_2,u_3+iu_4)^T=m\varphi(z)+\frac{1}{2}ez\overline{\varphi'(z)}+\overline{\psi(z)}, \tag{2}$$

$$Tu=\begin{pmatrix}(Tu)_2-i(Tu)_1\\ (Tu)_4-i(Tu)_3\end{pmatrix}=$$

$$=\frac{\partial}{\partial s(x)}\Big[(A-2E)\varphi(z)+Bz\overline{\varphi'(z)}+2\mu\overline{\psi(z)}\Big], \tag{3}$$

where $\varphi=(\varphi_1,\varphi_2)^T$ and $\psi=(\psi_1,\psi_2)^T$ are arbitrary analytic vector-functions, $\dfrac{\partial}{\partial s(x)}=$ $n_1\dfrac{\partial}{\partial x_2}-n_2\dfrac{\partial}{\partial x_1}$,

$$\begin{aligned}
(Tu)_1&=(a\theta'+c_0\theta'')n_1-(a_1\omega'+cw'')n_2-2\frac{\partial}{\partial s(x)}(\mu_1u_2+\mu_3u_4),\\
(Tu)_2&=(a\theta'+c_0\theta'')n_2+(a_1\omega'+cw'')n_1+2\frac{\partial}{\partial s(x)}(\mu_1u_1+\mu_3u_3),\\
(Tu)_3&=(c_0\theta'+b\theta'')n_1-(c\omega'+a_2w'')n_2-2\frac{\partial}{\partial s(x)}(\mu_3u_2+\mu_2u_4),\\
(Tu)_4&=(c_0\theta'+b\theta'')n_2+(c\omega'+a_2w'')n_1+2\frac{\partial}{\partial s(x)}(\mu_3u_1+\mu_2u_3);
\end{aligned}$$

$$\theta'=\operatorname{div}u',\quad \theta''=\operatorname{div}u'',\quad \omega'=\operatorname{rot}u'\quad \omega''=\operatorname{rot}u'',$$

$$A=2\mu m,\quad B=\mu e,\quad \mu=\begin{bmatrix}\mu_1&\mu_3\\ \mu_3&\mu_2\end{bmatrix},\quad E=\begin{bmatrix}1&0\\0&1\end{bmatrix}.$$

μ_1, μ_2, μ_3, a_1, a_2, c, $a=a_1+b_1$, $b=a_2+b_2$, $c_0=c+d$, b_1, b_2 and d are elastic constants satisfy the certain conditions [2].

(2) and (3) are analogous to the Kolosov-Muskhelishvili's formulas for the linear theory of eslastic mixture.

2. Let D denoted the upper half-plane. Clearly the boundary of D is ox_1 axis. Let $L=]-\infty,\infty[$ and choose the exterior unit normal $n=(n_1,n_2)^T=(0,-1)^T$.

The boundary value problems under consideration can be formulated in the form; Find a regular solution to the system (1) in D satisfying one of the following boundary conditions:

$$(u_2,u_4)^T=\overset{(1)}{f}_0(x_1),\quad ((Tu)_1,(Tu)_3)^T=\overset{(1)}{F}_0(x_1),\quad x_1\in L, \tag{4}$$

$$(u_1,u_3)^T=\overset{(2)}{f}_0(x_1),\quad ((Tu)_2,(Tu)_4)^T=\overset{(2)}{F}_0(x_1),\quad x\in L, \tag{5}$$

where $\overset{(j)}{f}_0$ and $\overset{(j)}{F}_0$, $j=1,2$, are given vector-functions satisfying certain smoothness conditions and also some conditions at infinity $\overset{(j)}{f}_0=\overset{(j)}{\alpha}+\overset{(j)}{\beta}/|x_1|^{1+\nu}$, $\overset{(j)}{F}_0=\overset{(j)}{\gamma}+\overset{(j)}{\delta}/|x_1|^{1+\nu}$, where $\overset{(j)}{\alpha}$, $\overset{(j)}{\beta}$, $\overset{(j)}{\gamma}$ and $\overset{(j)}{\delta}$, $j=1,2$ are an arbitrary real constant vectors, and $\nu>0$.

Using the Green formula [1] it is easy to prove.

Theorem 1. *The general solution of the homogeneous problem* (4) *is represented by the formula $U=\beta$, where β is an arbitrary real constant vector.*

Theorem 2. *The general solution of the homogeneous problem* (5) *is represented by the formula $U=i\delta$, where δ is an arbitrary real constant vector.*

Let now $(Tu)_k=\dfrac{\partial W_k}{\partial s}$, $k=\overline{1,4}$. Then (4) and (5) boundary conditions can be rewritten in the form:

$$(u_2, u_4)^T = \overset{(1)}{f_0}(t), \quad (W_1, W_3)^T = \overset{(1)}{\chi}(t) + \overset{(1)}{e}, \quad t \in L, \tag{6}$$

$$(u_1, u_3)^T = \overset{(2)}{f_0}(t), \quad (W_2, W_4)^T = \overset{(2)}{\chi}(t) + \overset{(2)}{e}, \quad t \in L, \tag{7}$$

where $\overset{(1)}{\chi}(t) = \int_{-\infty}^{t} \overset{(j)}{F_0}(t_0)dt_0$, $t \in L$, $\overset{(1)}{e}$ and $\overset{(2)}{e}$ are an arbitrary real constant vectors.

3. On the basis of (2) and (3), the problem (6) is reduced to finding two analytic vector-functions $\varphi(z)$ and $\psi(z)$ in the D by the boundary conditions

$$\operatorname{Re}(i\varphi(t)) = -\mu\overset{(1)}{f_0}(t) - \frac{1}{2}\overset{(1)}{\chi}(t) - \frac{1}{2}\overset{(1)}{e} = f(t) - \frac{1}{2}\overset{(1)}{e}, \quad t \in L, \tag{8}$$

$$\begin{aligned}\operatorname{Re}(i\psi(t)) &= \overset{(1)}{f_6}(t) + mf(t) - \frac{1}{2}etf'(t) - \frac{1}{2}m\overset{(1)}{e} \\ &= F(t) - \frac{1}{2}m\overset{(1)}{e}, \quad t \in L.\end{aligned} \tag{9}$$

The boundary condition (8) and (9) are the Riemann-Hilbert problem in the privan case for domain D.

A solution of the problems can be represented in the form [3]

$$\varphi(z) = -\frac{1}{\pi}\int_L \frac{f(t)dt}{t-z} + \frac{i}{2}\overset{(1)}{e}, \quad \psi(z) = -\frac{1}{\pi}\int_L \frac{F(t)dt}{t-z} + \frac{i}{2}m\overset{(1)}{e}, \tag{10}$$

Substituting (10) in (2), we obtain

$$U(z) = -\frac{m}{n}\int_L \frac{f(t)dt}{t-z} - \frac{ez}{2\pi}\int_L \frac{f'(t)dt}{t-\overline{z}} - \frac{1}{\pi}\int_L \frac{F(t)dt}{t-\overline{z}}. \tag{11}$$

(11) represent a regular solution of the problem (4).

4. By virtue of formulas (2) and (3), the boundary conditions (7) take the form

$$\operatorname{Re}\varphi(t) = \mu\overset{(2)}{f_0} - \frac{1}{2}\overset{(2)}{\chi}(t) - \frac{1}{2}\overset{(2)}{e} = g(t) - \frac{1}{2}\overset{(2)}{e}, \quad t \in L, \tag{12}$$

$$\operatorname{Re}\psi(t) = \overset{(2)}{f_0}(t) - mg(t) - \frac{1}{2}etg'(t) + \frac{1}{2}m\overset{(2)}{e} = h(t) + \frac{1}{2}m\overset{(2)}{e}, \quad t \in L, \tag{13}$$

The solution of problems (12) and (13) can be represented as [3]

$$\varphi(z) = \frac{1}{\pi i}\int_L \frac{g(t)dt}{t-z} - \frac{1}{2}\overset{(2)}{e}, \quad \psi(z) = \frac{1}{\pi i}\int_L \frac{h(t)dt}{t-z} + \frac{1}{2}m\overset{(2)}{e}. \tag{14}$$

From (14) and (2) we readily obtain

$$U(x) = \frac{m}{\pi i}\int_L \frac{g(t)dt}{t-z} - \frac{ez}{2\pi i}\int_L \frac{g'(t)dt}{t-\overline{z}} - \frac{1}{\pi i}\int_L \frac{h(t)dt}{t-\overline{z}}. \tag{15}$$

(15) is a regular solution of the problem (5).

R E F E R E N C E S

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