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ON THE ASYMPTOTICAL BEHAVIOR OF NEUTRONS PHASE DENSITY IN THE CASE AN ISOTROPIC POINT SOURCE

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Abstract. We investigate the asymptotical behavior of the phase density of neutrons near of the source in the problem for the isotropic point source.

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We are interested the phase density of the neutrons emitted from single power source which is disposed on the plane and radiating neutrons to the direction $\mu = \mu_0$ with length of the wave $\lambda = \lambda_0$. To this end, we must seek the solution of the equation [1]

$$\mu \frac{\partial G(x_0, \mu_0, \lambda_0 \to x, \mu, \lambda)}{\partial x} + G = \int_a^\lambda \int_{-1}^{+1} K(\lambda, \lambda') G(x_0, \mu_0, \lambda_0 \to x, \mu', \lambda') d\mu' d\lambda',$$

$$x \in (-\infty, +\infty), \quad \mu \in [-1, +1], \quad \lambda \in [a, b]$$

satisfying the boundary condition

$$2\pi\mu(G(x_0^+, \mu_0, \lambda_0 \to x, \mu, \lambda) - G(x_0^-, \mu_0, \lambda_0 \to x, \mu, \lambda)) = \delta(\mu - \mu_0)\delta(\lambda - \lambda_0) \tag{1}$$

and satisfying also the addition condition on the infinity

$$\lim_{|x| \to \infty} G(x_0, \mu_0, \lambda_0 \to x, \mu, \lambda) = 0. \tag{2}$$

Here δ is the Dirac function.

In order that satisfy the condition (2) we shall seek the solution on the form

$$G = \int_a^\lambda \int_0^1 u(\nu, \zeta) \exp(-\frac{x - x_0}{\nu}) \varphi_{\nu,(\zeta)}(\mu, \lambda) d\nu d\zeta, \qquad x > x_0 \tag{3}$$

$$G = -\int_a^\lambda \int_{-1}^0 u(\nu, \zeta) \exp(-\frac{x - x_0}{\nu}) \varphi_{\nu,(\zeta)}(\mu, \lambda) d\nu d\zeta, \qquad x < x_0 \tag{4}$$

where $\varphi_{\nu,(\zeta)}(\mu, \lambda)$, $\nu \in [-1, +1]$, $\zeta \in [a, b]$, is the singular eigenfunction of the characteristic equation

$$(\nu - \mu)\varphi_\nu(\mu, \lambda) = \nu \int_a^\lambda \int_{-1}^{+1} K(\lambda, \lambda') \varphi_\nu(\mu', \lambda') d\mu' d\lambda'$$

which is normalized as follows

$$\int_a^b \int_{-1}^1 \varphi_{\nu,(\zeta)}(\mu, \lambda) d\mu d\lambda = 1$$

and represented in the following form [2]

$$\varphi_{\nu,(\zeta)}(\mu,\lambda)=\frac{K(\lambda,\zeta)\theta(\lambda-\zeta)}{\nu-\mu}+(\delta(\zeta-\lambda)-\int_{-1}^{+1}\frac{K(\lambda,\zeta)\theta(\lambda-\zeta)}{\nu-\mu'}d\mu')\delta(\nu-\mu).$$

Here θ is the Heaviside function. In representation of G the function $u(\nu,\zeta)$ is the unknown function and our primary task is to construct u. When $x\to x_0$ then from (3) and (4) we obtain

$$G=\int_a^{\lambda}\int_0^1 u(\nu,\zeta)\varphi_{\nu,(\zeta)}(\mu,\lambda)d\nu d\zeta,\qquad x>x_0$$

$$G=-\int_a^{\lambda}\int_{-1}^0 u(\nu,\zeta)\varphi_{\nu,(\zeta)}(\mu,\lambda)d\nu d\zeta,\qquad x<x_0.$$

From condition (1) for the u we obtain

$$\delta(\mu-\mu_0)\delta(\lambda-\lambda_0)=2\pi\mu\int_a^{\lambda}\int_{-1}^1 u(\nu,\zeta)\varphi_{\nu,(\zeta)}(\mu,\lambda)d\nu d\zeta.$$

It is known that the set of singular eigenfunctions is the complete system [2]. In other words we have the next theorem

Theorem. *The arbitrary continuous function $\psi(\mu,\lambda)$, $\mu\in[-1+1]$, $\lambda\in[a,b]$ satisfying H^* [3] condition with respect to μ admits representation in the form*

$$\psi(\mu,\lambda)=\int_a^{\lambda}\int_{-1}^1 v(\nu,\zeta)\varphi_{\nu,(\zeta)}(\mu,\lambda)d\nu d\zeta.$$

Besides, it is truth the following equality

$$\int_a^b\int_{-1}^1 \mu\varphi_{\nu,(\zeta)}(\mu,\lambda)\tilde{\varphi}^*_{\nu',(\zeta')}(\mu,\lambda)d\mu d\lambda=\delta(\nu-\nu')\delta(\zeta-\zeta'),\tag{5}$$

where

$$\tilde{\varphi}^*_{\nu,(\zeta)}(\mu,\lambda)=\varphi^*_{\nu,(\zeta)}(\mu,\lambda)+\int_{\lambda}^{\zeta}\varphi^*_{\nu,(\zeta')}(\mu,\lambda)g(\nu,\zeta,\zeta')d\zeta'$$

and

$$\varphi^*_{\nu,(\zeta)}(\mu,\lambda)=\frac{K(\zeta,\lambda)\theta(\zeta-\lambda)}{\nu-\mu}+(\delta(\zeta-\lambda)-\int_{-1}^{+1}\frac{K(\zeta,\lambda)\theta(\zeta-\lambda)}{\nu-\mu'}d\mu')\delta(\nu-\mu),$$

$\varphi^*_{\nu,(\zeta)}(\mu,\lambda)$, $\nu\in[-1,+1]$, $\zeta\in[a,\lambda]$, is the singular eigenfunction of the characteristic equation

$$(\nu-\mu)\varphi^*_{\nu}(\mu,\lambda)=\nu\int_{\lambda}^b\int_{-1}^{+1}K(\lambda',\lambda)\varphi^*_{\nu}(\mu',\lambda')d\mu' d\lambda',$$

which is normalized as follows

$$\int_a^b\int_{-1}^1\varphi^*_{\nu,(\zeta)}(\mu,\lambda)d\mu d\lambda=1$$

and represented in the following form

$$\varphi^*_{\nu,(\zeta)}(\mu,\lambda)=\frac{K(\zeta,\lambda)\theta(\zeta-\lambda)}{\nu-\mu}+(\delta(\zeta-\lambda)-\int_{-1}^{+1}\frac{K(\zeta,\lambda)\theta(\zeta-\lambda)}{\nu-\mu'}d\mu')\delta(\nu-\mu).$$

Here $g(\nu,\zeta,\zeta')$ is unique solution of the equation

$$g(\nu,\zeta,\zeta')-\int_{\zeta'}^{\zeta}S(\nu,\zeta",\zeta')g(\nu,\zeta,\zeta")d\zeta"=S(\nu,\zeta,\zeta'),$$

where

$$S(\nu,\zeta,\zeta')=2\int_{-1}^{1}\frac{\nu K(\zeta,\zeta')}{\nu-\mu}d\mu$$

$$-\int_{\zeta'}^{\zeta}\int_{-1}^{1}\frac{\nu K(\lambda,\zeta')}{\nu-\mu}d\mu\int_{-1}^{1}\frac{\nu K(\zeta,\lambda)}{\nu-\mu}d\mu d\lambda-\int_{\zeta'}^{\zeta}K(\lambda,\zeta')K(\zeta,\lambda)d\lambda,$$

$$a\le\zeta'\le\zeta\le b.$$

It follows from the preceding theorem and equation (5) that

$$v=\tilde{\varphi}^*_{\nu,(\zeta)}(\mu_0,\lambda_0).$$

Therefore, now we can write

$$G(x_0,\mu_0,\lambda_0\to x,\mu,\lambda)=\frac{1}{2\pi}\int_a^{\lambda}\int_0^1\varphi_{\nu,(\zeta)}(\mu,\lambda)\tilde{\varphi}^*_{\nu,(\zeta)}(\mu_0,\lambda_0)d\nu d\zeta,\qquad x>x_0$$

and

$$G(x_0,\mu_0,\lambda_0\to x,\mu,\lambda)=\frac{1}{2\pi}\int_a^{\lambda}\int_{-1}^0\varphi_{\nu,(\zeta)}(\mu,\lambda)\tilde{\varphi}^*_{\nu,(\zeta)}(\mu_0,\lambda_0)d\nu d\zeta,\qquad x<x_0.$$

If we apply the normalization condition for $\varphi_{\nu,(\zeta)}$ and $\varphi^*_{\nu,(\zeta)}$ then for of the neutrons density

$$\rho(x_0\to x)=2\pi\int_a^b\int_{-1}^1\int_a^b\int_{-1}^1G(x_0,\mu_0,\lambda_0\to x,\mu,\lambda)d\mu_0d\lambda_0d\mu d\lambda$$

we can write

$$\rho(x_0\to x)=\int_0^1\exp(-\mid x-x_0\mid/\nu)R(\nu)d\nu,$$

where

$$R(\nu)=1+\int_a^b\int_a^b g(\nu,\zeta,\zeta')d\zeta'd\zeta.$$

Applying the same procedure as in [3,§5.4] for $\rho_{pt}(x_0\to x)$, where

$$\rho_{pt}(x_0\to x)=-\frac{1}{2\pi(x-x_0)}\frac{d}{dx}\rho(x_0\to x),$$

for the small $(x - x_0)$ we obtain

$$\rho_{pt}(x_0 \to x) \approx \frac{1}{4\pi(x - x_0)^2}.$$

Thus, obtained the result for the multivelocity case is identical to the result from [3, §5.4] for the onevelocity case. This was expected.

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