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ON THE HOMOLOGY THEORY OF THE CLOSED GEODESIC PROBLEM

Saneblidze S.

Abstract. In this paper we investigate a problem from differential geometry: how many closed geodesics lie on a closed, compact, Riemannian manifold.

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Let X be a closed, compact, Riemannian manifold. A problem in differential geometry is: how many closed geodesics lie on X? To relate this problem with a topological problem one considers the free loop space ΛX, all smooth maps from the circle S^1 into X, and the energy functional $E : \Lambda X \to \mathbb{R}$ defined by

$$E(f) = \int_{S^1} \langle f'(s), f'(s) \rangle ds, \quad f \in \Lambda X, \ \langle \ , \ \rangle \text{ is the Riemannian metric on } X.$$

Closed geodesics on X are the critical points of the function E. The above question becomes more subtle when the fundamental group of X is finite, in particular, is trivial. Using an infinite dimensional Morse theory Gromoll and Meyer proved the following theorem in 1969:

Theorem 1. *Let X be a simply connected closed compact smooth manifold of dimension greater than 1. If the Betti numbers $\beta_i(\Lambda X; \mathbb{R})$ grow unbounded, then X has infinitely many geometrically distinct closed geodesics in any Riemannian metric.*

Since the arguments of the theorem work for the Betti numbers $\beta_i(\Lambda X; \Bbbk)$ with respect to any coefficient field $\Bbbk$, this result has motivated a question, the 'closed geodesic problem,' to find simple criteria in terms of the cohomology algebra $H^*(X; \Bbbk)$ of X which imply that the Betti numbers $\beta_i(\Lambda X; \Bbbk)$ are unbounded. The space ΛX is homotopic to the space of all continuous maps $S^1 \to X$, so that we denote the latter space again by ΛX; moreover, we can also assume in the sequel X to be a finite CW-complex. Let $\Bbbk$ be a commutative ring with unit. Assume that the i^{th}-cohomology group $H^i(\Lambda X; \Bbbk)$ of ΛX is finitely generated as a $\Bbbk$-module and refer to the cardinality of a minimal generating set of $H^i(\Lambda X; \Bbbk)$, denoted by $\beta_i(\Lambda X; \Bbbk)$, as the *generalized i^{th}-Betti number of* ΛX. Below we state the aforementioned criteria in its most general form in the following

Theorem 2. *Let X be a simply connected space and $\Bbbk$ a commutative ring with unit. If $H^*(X; \Bbbk)$ is finitely generated as a $\Bbbk$-module and $H^*(\Lambda X; \Bbbk)$ has finite type, then the generalized Betti numbers $\beta_i(\Lambda X; \Bbbk)$ grow unbounded if and only if $\tilde{H}^*(X; \Bbbk)$ requires at least two algebra generators.*

Theorem 2 was proved by Sullivan and Vigué-Poirrier when $\Bbbk$ is a field of characteristic zero, and then it was conjectured for $\Bbbk$ to be a field of positive characteristic: A simple example provided by the Stiefel manifold $V_2(\mathbb{R}^{2n+1})$ shows that a manifold may have the rational cohomology with one algebra generator, but having infinitely many closed geodesics; note also that the cohomology $H^*(V_2(\mathbb{R}^{2n+1});\mathbb{Z}_2)$ has two algebra generators. Since then a number of papers deals with this conjecture but it remained to be open even for X to be a finite CW-complex and $\Bbbk$ a finite field. We prove Theorem 2. The proof relies on a recently constructed by author small algebraic model

$$(\bar{V}_{\Bbbk},\bar{d}_h)\xleftarrow{\pi}(RH\otimes\bar{V}_{\Bbbk},d_\omega)\xleftarrow{\iota}(RH_{\Bbbk},d_h)$$

of the free loop fibration

$$\Omega X\to\Lambda X\to X$$

consisting of differential graded algebras (dga's) in which ι and π are the standard inclusion and projection respectively. Namely, $(RH,d_h)\to C^*(X;\mathbb{Z})$ is a filtered homotopy G-algebra(hga) model of $C^*(X;\mathbb{Z})$, the singular cochain complex of X, with $H=H^*(X;\mathbb{Z})$, $RH_{\Bbbk}=RH\otimes_{\mathbb{Z}}\Bbbk$, $RH_{\Bbbk}=T(V_{\Bbbk})$, $V_{\Bbbk}=V\otimes_{\mathbb{Z}}\Bbbk$, $\bar{V}$ is the desuspension of $V,\bar{V}=s^{-1}V\oplus\Bbbk$, and the differential $\bar{d}_h$ is obtained by restriction of d_h to V. The hga structure on RH in particular means the existence of a binary operation $E_{1,1}:RH\otimes RH\to RH$ satisfying conditions similar to Steenrod's cochain $\smile_1$-operation:

$$dE_{1,1}(a;b)-E_{1,1}(da;b)+(-1)^{|a|}E_{1,1}(a;db)=(-1)^{|a|}a\cdot b-(-1)^{|a|(|b|+1)}b\cdot a,$$

so it measures the non-commutativity of the $\cdot$ product on RH. Using the above model we construct two infinite sequences in $H^*(\Lambda X;\Bbbk)$ and take all possible products of their components to detect a submodule of $H^*(\Lambda X;\Bbbk)$ at least as large as the polynomial algebra $\Bbbk[x,y]$ that implies the proof of Theorem 2.

An idea of constructing such sequences comes from a recent paper of the author in which an analogous result is established for the (based) loop space ΩX. In turn, the aforementioned sequences can be thought of as a certain generalization of W. Browder's notion of ∞-implications. Note that Browder's ∞-implications are used by J. McCleary to detect a submodule of $H^*(\Omega X;\Bbbk)$ isomorphic to the polynomial algebra $\Bbbk[x,y]$ for $\Bbbk$ a field. On the one hand, a difficulty to lift a sequence from $H^*(\Omega X;\Bbbk)$ into $H^*(\Lambda X;\Bbbk)$ is because of a canonical homomorphism $H^*(\Lambda X;\Bbbk)\to H^*(\Omega X;\Bbbk)$ fails to be a surjection. On the other hand, it is impossible to find a sequence in $H^*(\Omega X;\Bbbk)$ consisting of iterated powers of some element in $H^*(\Omega X;\Bbbk)$ for $\Bbbk$ to be a finite field.

Our construction of sequences in $H^*(\Lambda X;\Bbbk)$ heavily uses the explicit formula for the product on the complex $(RH\otimes\bar{V}_{\Bbbk},d_\omega)$. Given a dga (A,d) isomorphic to the tensor algebra $T(V)$, the complex $(A\otimes\bar{V},d_\omega)$ was in fact known as a reduction of the Hochschild chain complex of (A,d) that describes the additive structure of $H^*(\Lambda X;\Bbbk)$ when $A=C^*(X;\Bbbk)$. Namely, the differential d_ω is given by

$$d_\omega(u\otimes\bar{a})=du\otimes\bar{a}-(-1)^{|u|}(1\otimes s^{-1})\chi(u\otimes da)-(-1)^{|u|+|a|}(ua-(-1)^{|a||u|}au)\otimes 1,$$

in which $\chi:A\otimes A\to A\otimes V$ is a map defined by

$$\chi(u\otimes a)=\begin{cases}0, & u\otimes a=u\otimes 1,\\ u\otimes a_1, & u\otimes a=u\otimes a_1,\\ \sum_{1\le i\le p}(-1)^{\varepsilon}a_{i+1}\cdots a_p u a_1\cdots a_{i-1}\otimes a_i, & u\otimes a=u\otimes a_1\cdots a_p,\\ & \qquad p\ge 2,\end{cases}$$

$\varepsilon = (|a_{i+1}| + \cdots + |a_p|)(|u| + |a_1| + \cdots + |a_i|)$, $a_i \in V$. Here a new aspect is to introduce a multiplication on the above complex satisfying the Leibnitz rule and calculating the cohomology algebra $H^*(\Lambda X; \Bbbk)$. It is well known that the Hochschild chain complex of the dga $C^*(X; \Bbbk)$ has a canonical simplicial structure, but the induced product on it is not *geometric*, i.e., does not correspond to the product on $H^*(\Lambda X; \Bbbk)$. Instead we have detected an F_n-set structure on it, where F_n is a certain n-dimensional polytope, called the *freehedron*; this polytope can be obtained from the standard n-simplex Δ^n by truncations at the initial and terminal vertices. In particular, F_0 is a point, F_1 is an interval, F_2 is a pentagon, F_3 has eight 2-faces (4 pentagon and 4 quadrilateral), 18 edges and 12 vertices (see Fig. 1 below).

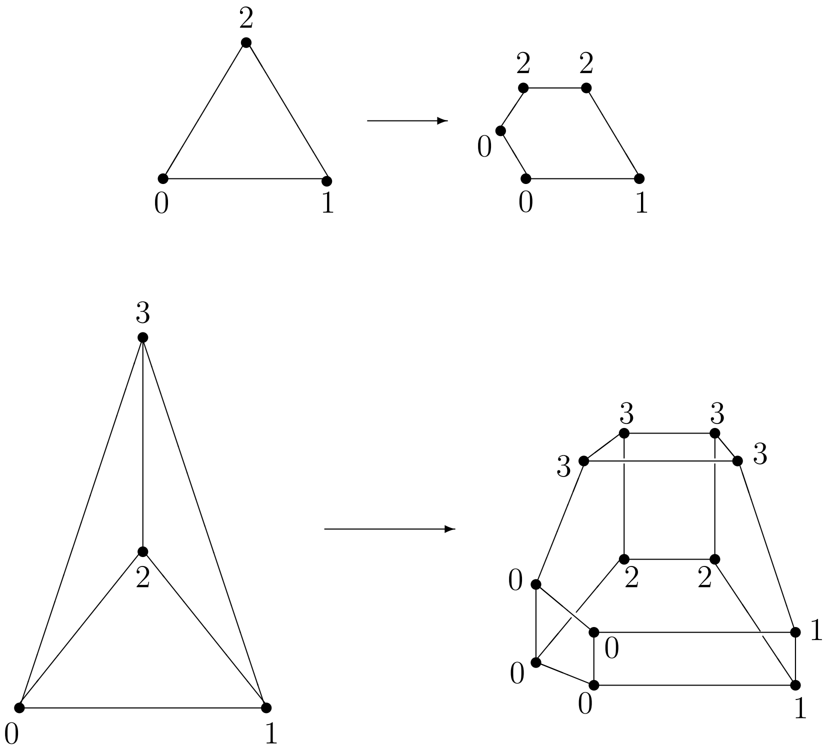

Fig. 1. The freehedra F_2 and F_3 obtained from Δ^2 and Δ^3 by truncations.

In general, the freehedra can be thought of as a combinatorial model of a free loop space. We have had defined an explicit diagonal Δ_F of F_n, and then by a combinatorial analysis of Δ_F have determined the required product on the Hochschild chain complex of $C^*(X; \Bbbk)$. Now we obtain the product on $(RH \otimes \bar{V}_{\Bbbk}, d_\omega)$, too. More precisely, denote $\smile_1 = E_{1,1}$ and for $u \otimes x, v \otimes y \in RH \otimes \bar{V}_{\Bbbk}$, let

$$
(u\otimes x)(v\otimes y)=\begin{cases}
uv\otimes 1, & (x,y)=(1,1),\\[2ex]
uv\otimes \bar b+(-1)^{(|v|+1)(|b|+1)}(u\smile_1 b)v\otimes 1, & (x,y)=(1,\bar b),\\[2ex]
(-1)^{(|a|+1)|v|}uv\otimes \bar a+u(a\smile_1 v)\otimes 1, & (x,y)=(\bar a,1),\\[2ex]
(-1)^{(|a|+1)|v|}uv\otimes \overline{a\smile_1 b} & \\
+u(a\smile_1 v)\otimes \bar b & \\
+(-1)^{|a|(|v|+|b|+1)+|v||b|}(u\smile_1 b)v\otimes \bar a & \\
+(-1)^{(|a|+|v|)(|b|+1)}(u\smile_1 b)(a\smile_1 v)\otimes 1 & \\
-\sum(-1)^{\epsilon_1}u(a_1\smile_1 v)\otimes \overline{a_2\smile_1 b} & \\
+\sum(-1)^{\epsilon_2}(u\smile_1 b)(a_1\smile_1 v)\otimes \bar a_2 & \\
+\sum(-1)^{\epsilon_3}(u\smile_1 b_2)v\otimes \overline{a\smile_1 b_1} & \\
+\sum(-1)^{\epsilon_4}(u\smile_1 b_2)(a\smile_1 v)\otimes \bar b_1, & (x,y)=(\bar a,\bar b),
\end{cases}
$$

where $da=\sum a_1a_2$, $db=\sum b_1b_2$ and $\epsilon_1=|a_1||b|+(|a_2|+1)|v|$, $\epsilon_2=|a_2|(|v|+1)+(|a|+|v|)|b|$, $\epsilon_3=(|a|+|b_2|)(|v|+1)+(|a|+|b_1|)|b_2|$, $\epsilon_4=(|a|+|v|)(|b_2|+1)+(|b_1|+1)|b_2|$.

Finally, note that by means of the dga $(RH\otimes \bar V_{\Bbbk}, d_\omega)$ the first sequence is relatively easy to construct rather than the second one in $H^*(\Lambda X;\Bbbk)$. In particular, the construction of the second sequence requires to consider both primary and secondary cohomology operations on $H^*(X;\Bbbk)$.

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Author's address:

S. Saneblidze
A. Razmadze Mathematical Institute
Dep. of Geometry and Topology
1, M. Aleksidze St., Tbilisi 0193
Georgia
E-mail: sane@rmi.acnet.ge