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ON THE FUCHSIAN SYSTEMS FREE FROM ACCESSORY PARAMETERS

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Abstract. We consider a system of Fuchsian linear differential equations free from accessory parameters with 3 singular points and study monodromy groups of such systems.

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It is well known that for any homomorphism

$$\chi : \pi_1(CP^1 - D, z_0) \to GL(p, C) \tag{1}$$

of the fundamental group of the complement of a set $D = \{a_1, ..., a_n\}$ of points of the Riemann sphere CP^1 into the group of complex-valued nondegenerate matrices of order p one can construct a Fuchsian equation

$$y^{(p)} + q_1(z)y^{(p-1)} + ... + q_p(z)y = 0 \tag{2}$$

with given monodromy (1), whose set D' of singular points coincides with the set $D \cup \{b_1, ..., b_m\}$. The additional singular points $\{b_1, ..., b_m\}$ do not contribute to the monodromy and are called false singular points. From (1), by a standard method we construct a vector bundle F' over $CP^1 - D$ with structure group $GL(p; C)$. Let F be Yu. Manins continuation of this bundle to all of CP^1. Then according to the Birkhoff-Grothendieck theorem we have $F \cong O(-k_1) \oplus O(-k_2) \oplus ... \oplus O(-k_p)$, where $k_1 \geq ... \geq k_p$, and $O(-r)$ is the rth power of the Hopf bundle $O(-1)$ on CP^1. Denote by l the number of the first numbers $k_1, ..., k_p$, $(k_1 = ... = k_l)$ that are equal to each other.

It is known, that for any irreducible representation (1) a Fuchsian equation (2) exists with given monodromy (1), the number m of additional false singular points of which satisfies the inequality $m \leq [(n-2)p(p-1)]/2 - \sum_{i=1}^{p}(k_1 - k_i) + 1 - l$.

Theorem 1. *[1] For any Fuchsian equation on the Riemann sphere it is possible to construct a Fuchsian system*

$$df = \sum_{i=1}^{p} \left(\frac{B_i}{z - a_i} dz \right) f$$

with the same singular points and the same monodromy.

While an arbitrary equivalence class of irreducible representations

$$\pi_1(C - \{0, 1\}) \to GL(2, C) \tag{3}$$

is induced by a certain hypergeometric differential equation

$$x(1-x)y'' + (c - (a+b+1)x)y' - aby(x) = 0, \tag{4}$$

and vice versa, there are some classes of reducible representations which are not induced by (4). In [2] the author sets up the canonical bases with respect to which the twenty reducible classes induced by (4) are realized in a simple form. This includes the necessary connection formulas in the degenerate cases (in which a or b or $c-a$ or $c-b$ is an integer).

Example. Let generators of the representation (3) are

$$G_1 = \begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 1 & s_2 \\ 0 & 1 \end{pmatrix}, \quad G_3 = \begin{pmatrix} 1 & -s_1 - s_2 \\ 0 & 1 \end{pmatrix},$$

where $s_1 s_2 (s_1 + s_2) \neq 0$, then does not exists fuchsian differential equation with 3 singular points and with representation generated by G_1, G_2, G_3.

Proposition 1. *For any representation in dimensional two, whose generators are different from G_1, G_2, G_3 realisably as monodromy representation of hypergeometric equations.*

Proposition 2. (See [1],[3].) *1) For $n = 3$ any irreducible representation (1) of dimension $p = 2$ can be realized as the monodromy of the Gauss equation, i.e., a second-order Fuchsian equation with three singular points.*

2) If the representation (1) is realized as the representation of the monodromy of the Fuchsian equation (2) without additional apparent singular points, then it is also realized as the representation of a monodromy of a certain Fuchsian system with the same singular points.

3) Let the representation (1) $(n > 2)$ be reducible and let each of the matrices G_i of the monodromy, corresponding to a circuit of the point a_i along a small loop, reduce to a Jordan cell. Then the Fuchsian equation (2) does not exist without additional false singular points, whose monodromy coincides with (1).

Proposition 3. [1] *For any three points on the Riemann sphere and for any irreducible representation (1) of dimension $p = 4$ there exists a Fuchsian system on CP^1 with given monodromy (1), whose singular points coincide with three given points.*

Example. The Fuchsian system constructed from Gauss equation

$$y'' + \frac{\gamma - (\alpha + \beta + 1)z}{z(1-z)} y' - \frac{\alpha\beta}{z(1-z)} y = 0$$

has the form

$$df = \left(\begin{pmatrix} 0 & 0 \\ -\alpha\beta & -\gamma \end{pmatrix} \frac{dz}{z} + \begin{pmatrix} 0 & 1 \\ 0 & \gamma - (\alpha + \beta) \end{pmatrix} \frac{dz}{z-1} \right) f.$$

Fuchsian systems of differential equations on CP^1, which are free from accessory parameters, have the following important property: their monodromies can be calculated explicitly from their coefficients (for an arbitrary system in general there is no way to calculate its monodromy) [4].

Considers the system of differental equations of the form

$$(xI_n - T)\frac{dY}{dx} = AY \tag{5}$$

on CP^1 of rank n, which called *Okubo normal form*, where $T = t_1 I_{n_1} \oplus ... \oplus t_p I_{n_p}$, $t_i \in C(1 \le i \le p)$, $t_i \ne t_j (i \ne j)$, $n_1 + ... + n_p = n$, A is a diagonalizable and $A \in End(n, C)$. The matrix A is decomposed into blocks of submatrices and the system is viewed as Fuchsian over CP^1 with regular singular points at $x = t_1, ..., t_p, \infty$. By special gauge transformation, $Y = PZ$, it is possible to determine all systems which are irreducible and free from accessory parameters, therefore there exists a new class of extensions of the Gauss hypergeometric function.

Let

$$(tI - B)\frac{dx}{dt} = Ax \tag{6}$$

is (5) type system. The following conditions for the coefficients of the system we assume:

(i) the matrix B is diagonal with eigenvalues $\lambda_1, ..., \lambda_p$ that have multiplicities $n_1, ..., n_p$ satisfying the inequalities $n_1 \ge ... \ge n_p$;

(ii) each diagonal block A_{ii} (in the same partition of A as B) is a diagonal matrix with distinct eigenvalues;

(iii) the matrix A is diagonalizable and it has eigenvalues $\rho_1, ..., \rho_q$ that have multiplicities $m_1, ..., m_q$ satisfying the relations $m_1 \ge ... \ge m_q$.

It is known, that the condition $n_1 + m_1 \le d$ (d is the rank of the system) is necessary for the system to be irreducible (a system is called irreducible if there is no transformation $P(t) \in GL(d, C(t))$ reducing the system to a system with a block triangular coefficient matrix). A classification of the above systems which are free of accessory parameters and satisfy the inequality $n_1 + m_1 \le d$ is presented in [5].

Consider particular case of the system (6), where $B = diag(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, A is diagonalizable and has nonresonant nonnegative eigenvalues $\rho_1 = \rho_2, \rho_3, \rho_4$. Moreover, A has a block form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11}, A_{22} are diagonalizable 2×2 matrices with nonresonant eigenvalues.

This equation is Fuchsian, has three singular points $\lambda_1, \lambda_2, \infty$ and its exponents at these points are as follows: $(a_{11}, a_{22}, 0, 0)$, $(0, 0, a_{33}, a_{44})$, $(\rho_1, \rho_2, \rho_3, \rho_4)$.

Proposition 4. *The system (6) in assumptions above is accessory parameter free.*

From this proposition follows, that it is possible to calculate its monodromy group in terms of the exponents (up to diagonal transformations) and obtain necessary and sufficient conditions of irreducibility for the monodromy group in terms of the exponents.

Theorem 2. *The $d = 4$ dimensional system*

$$dF = \left(I \otimes \begin{pmatrix} 0 & 0 \\ -\alpha\beta & -\gamma \end{pmatrix} \frac{dz}{z} + \begin{pmatrix} 0 & 1 \\ 0 & \gamma - (\alpha + \beta) \end{pmatrix} \otimes I \frac{dz}{z-1} \right) F$$

is free from accessory parameters and have the form (6).

Final remark. The notion of a local system on CP^1-{finite set of points} was introduced by Riemann in order to study the classical Gauss hypergeometric function, which he did by studying rank two local systems on CP^1-{three points}. His investigation was a stunning success, in large part because any such local system is rigid. For example, in [6] proved that a bundle of rank 2 is rigid (that is, $dimH^1(CP^1, End(E)) = 0$) if and only if it admits a connection which is holomorphic everywhere except at 3 points with at most logarithmic singularities. From the theorem 2 follows, that the notations of the papers [7],[8] it is possible to extend in the four dimensional case also.

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