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ITO TYPE FORMULA FOR POISSON ANTICIPATING INTEGRAL

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Abstract. The quadratic variation of the anticipating Skorokhod integral with respect of compensated Poisson martingale is computed and anticipative Ito type formula for the so-called an anticipative Poisson semimartingales in terms of anticipative Skorokhod integrals is derived.

Keywords and phrases: Stochastic derivative, Skorokhod integral, Ito's formula, anticipative Poisson semimartingale.

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In the anticipative case the Ito type formula was obtained by Ustunel [1] in the Wiener space for random fields $F(x,\omega)$. This fields are fast decreasing with respect to x and argument x is replaced by the so-called Ito's anticipative process (with respect to Wiener process). The general case was considered by Nualart and Pardoux [2]. In case when $F(t,x)$ (for any x) is adapted diffusion process and x is replaced by Ito's anticipative process the anticipative Ito-Ventsel type formula was established by Martias [3]. The case where both $F(t,x,\omega)$ (for any x) and u_t are Ito's anticipative processes the Ito-Ventsel type formula and an integral variant of the Ito-Ventsel formula was obtained by Purtukhia ([4],[5]). In the Poisson case the similar questions was studied by Peccati and Tudor [6] and anticipative Ito type formula was established in terms of nonanticipative Ito integrals. Our aim is to derive anticipative Ito type formula for the so-called an anticipative Poisson semimartingales in terms of anticipative Skorokhod integrals [7].

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]})$ be a filtered probability space satisfying the usual conditions. Suppose that N_t is the standard Poisson process ($P(N_t = k) = \frac{t^k e^{-t}}{k!}, k = 0, 1, 2, ...$) and $\mathcal{F}_t$ is generated by $N(\mathcal{F}_t = \mathcal{F}_t^N), \mathcal{F} = \mathcal{F}_T$. Let M_t be the compensated Poisson process ($M_t = N_t - t$). Denote by $D_{\cdot}^M G$ the stochastic derivative of functional G (see Definition 4.1 [8]). In what follows we shall write $D.G$ instead of $D_{\cdot}^M G$.

For any integer $k \geq 1$ we introduce the seminorm

$$||F||_{2,k} = ||F||_{L_2(\Omega)} + \sum_{i=1}^{k} ||D^i F||_{L_2([0,T]^i\times\Omega)}$$

and denote by $D_{2,k}^M$ the completion of class of differentiable random variables with respect to the norm $||\cdot||_{2,k}$.

Definition 1. We denote by $L_{2,1}^M$ the class of processes $u \in L_2([0,T]\times\Omega)$ such that $u_t \in D_{2,1}^M$ for a.a. t and there exists a measurable version of $D_s u_t \in L_2([0,T]^2 \times \Omega)$.

Definition 2. We denote by $L_{2,2}^M$ the class of processes $u \in L_2([0,T]\times\Omega)$ such that $u_t \in D_{2,2}^M$ for a.a. t and there exists a measurable version of $D_r D_s u_t \in L_2([0,T]^3 \times \Omega)$.

Let $\Pi^n, n \in N$ be a sequence of partitions of the segment $[0,T]$ of the form $\Pi^n = \{0 = t_{0,n} < t_{1,n} < \cdots < t_{n,n} = T\}$ such that $|\Pi^n| = sup_k(t_{k+1,n} - t_{k,n}) \to 0$, as $n \to \infty$. In what follows we shall write (t_k) instead of $t_{k,n}$.

Proposition. *Let $\xi_t, t \in [0,T]$ be a measurable process such that $\xi \in L_2([0,T]\times\Omega)$. Then*

$$\sum_{k=0}^{n-1}\Big(\frac{1}{t_{k+1,n} - t_k, n}\int_{t_{k,n}}^{t_{k+1,n}} \xi_s ds\Big)(M_{t_{k+1,n}} - M_{t_k,n})^2 \to \int_0^T \xi_s ds$$

in $L_1(\Omega)$, as $n \to \infty$.

Proof. Let's enter the following designations:

$$\xi^m := \sum_{i=0}^{m-1}\left(\frac{1}{t_{i+1,m} - t_i, m}\int_{t_{i,m}}^{t_{i+1,m}} \xi_s ds\right) I_{[t_{i,m},t_{i+1,m}[},$$

$$\alpha_n(\xi) := \sum_{k=0}^{n-1}\left(\frac{1}{t_{k+1,n} - t_k, n}\int_{t_{k,n}}^{t_{k+1,n}} \xi_s ds\right)(M_{t_{k+1,n}} - M_{t_k,n})^2,$$

$$\alpha_n(\xi_m) := \sum_{i=0}^{m-1}\left(\frac{1}{t_{i+1,m} - t_i, m}\int_{t_{i,m}}^{t_{i+1,m}} \xi_s ds\right) I_{[t_{i,m},t_{i+1,m}[}.$$

Using the Cauchy-Bunyakovski inequality, it is not difficult to see that

$$E|\alpha_n(\xi)| \le \left\{E\sum_{k=0}^{n-1}\frac{(M_{t_{k+1,n}} - M_{t_k,n})^4}{t_{k+1,n} - t_k, n}\right\}^{1/2}\left\{E\sum_{k=0}^{n-1}\frac{(\int_{t_{k,n}}^{t_{k+1,n}}|\xi_s|ds)^2}{t_{k+1,n} - t_k, n}\right\}^{1/2}$$

$$\le C||\xi||_{L_2([0,T]\times\Omega)}.$$

Hence, we can write

$$E|\alpha_n(\xi) - \int_0^T \xi_s ds| \le E|\alpha_n(\xi - \xi^m)| + E|\alpha_n(\xi^m) - \int_0^T \xi_s^m ds|$$

$$+E\int_0^T|\xi - \xi_s^m|ds| \le E|\alpha_n(\xi^m) - \int_0^T \xi_s^m ds| + (C+1)||\xi - \xi^m||_{L_2([0,T]\times\Omega)}.$$

It is obvious that $\xi^m \to \xi$ in $L_2([0,T]\times\Omega)$ as $m \to \infty$. On the other hand, it is evident that for any fixed $m : \alpha_n(\xi^m) \to \int_0^T \xi_s^m ds$ in probability as $n \to \infty$. Moreover, due to the Holder's inequality, we can easily obtain that for any $p \in (1,2)$:

$$||\alpha_n(\xi^m)||_{L_p(\Omega)} \le C_p||\xi^m||_{L_2([0,T]\times\Omega)}.$$

Therefore, for each m the sequence of random variables $\{\alpha_n(\xi^m), n \in N\}$ is uniformly integrable, which with the convergence in probability implies that $\alpha_n(\xi^m) \to \int_0^T \xi_s^m ds$ in $L_1(\Omega)$ as $n \to \infty$. Passing now to the limit in above relation at first as $m \to \infty$, and after as $n \to \infty$ we complete the proof of the Proposition.

Theorem 1. *Let* $u \in L^M_{2,2}$. *Then*

$$\sum_{k=0}^{n-1}\left(\int_{t_k}^{t_{k+1}} u_s \delta M_s\right)^2 \to \int_0^T u_s^2 ds$$

in $L_1(\Omega)$, *as* $n \to \infty$.

Proof. Let $u, v \in L^M_{2,1}$. By virtue of the Cauchy-Bunyakovski inequality we can write

$$E\sum_{k=0}^{n-1}\Big|\left(\int_{t_k}^{t_{k+1}} u_s \delta M_s\right)^2 - \sum_{k=0}^{n-1}\left(\int_{t_k}^{t_{k+1}} v_s \delta M_s\right)^2\Big|$$

$$\leq \left(E\sum_{k=0}^{n-1}\left(\int_{t_k}^{t_{k+1}} (u_s - v_s)\delta M_s\right)^2\right)^{1/2}\left(E\sum_{k=0}^{n-1}\left(\int_{t_k}^{t_{k+1}} (u_s + v_s)\delta M_s\right)^2\right)^{1/2}.$$

Define u^n as follow:

$$u^n = \sum_{k=0}^{n-1} \overline{u}_k I_{[t_k, t_{k+1}[},$$

where

$$\overline{u}_{k,n} = \frac{1}{t_{k+1} - t_k}\int_{t_k}^{t_{k+1}} u_s ds$$

for $0 \leq k < n-1$ and $\overline{u}_{-1,n} = \overline{u}_{0,n} = 0$.

Substituting now $v = u^n$ in the above estimate, one can conclude that

$$E\sum_{k=0}^{n-1}\Big|\left(\int_{t_k}^{t_{k+1}} u_s \delta M_s\right)^2 - \sum_{k=0}^{n-1}\left(\int_{t_k}^{t_{k+1}} u_s^n \delta M_s\right)^2\Big| \to 0,$$

as $n \to \infty$.

On the other hand, due to the Proposition 3.2 [9], we have

$$\sum_{k=0}^{n-1}\left(\int_{t_k}^{t_{k+1}} u_s^n \delta M_s\right)^2 = \sum_{k=0}^{n-1}\left(\int_{t_k}^{t_{k+1}} \overline{u}_{k,n} \delta M_s\right)^2$$

$$= \sum_{k=0}^{n-1}\left(\frac{1}{t_{k+1} - t_k}\int_{t_k}^{t_{k+1}}\left(\int_{t_k}^{t_{k+1}} u_s ds\right)\delta M_s\right)^2$$

$$= \sum_{k=0}^{n-1}\left\{\frac{1}{t_{k+1} - t_k}\left[\left(M_{t_{k+1}} - M_{t_k}\right)\int_{t_k}^{t_{k+1}} u_s ds\right.\right.$$

$$\left.\left. - \int_{t_k}^{t_{k+1}}\left(\int_{t_k}^{t_{k+1}} D_r u_s ds\right) dr - \int_{t_k}^{t_{k+1}}\left(\int_{t_k}^{t_{k+1}} D_r u_s ds\right)\delta M_r\right]\right\}^2 :$$

$$:= \sum_{k=0}^{n-1}\left(a_{k,n}^2 - 2a_{k,n}b_{k,n} + b_{k,n}^2\right),$$

where

$$a_{k,n} = \frac{M_{t_{k+1}} - M_{t_k}}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} u_s ds,$$

$$b_{k,n} = \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} \left(\int_{t_k}^{t_{k+1}} D_r u_s ds \right) dr - \int_{t_k}^{t_{k+1}} \left(\int_{t_k}^{t_{k+1}} D_r u_s ds \right) \delta M_r.$$

Using the Cauchy-Bunyakovski inequality and the elementary inequality $(x+y)^2 \leq 2x^2 + 2y^2$, we can write that

$$E \sum_{k=0}^{n-1} b_{k,n}^2 \leq 2E \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} |D_r u_s|^2 ds dr$$

$$+2E \sum_{k=0}^{n-1} \left[\int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} |D_r u_s|^2 ds dr + \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} |D_\theta D_r u_s|^2 ds dr d\theta \right].$$

Hence, $\sum_{k=0}^{n-1} b_{k,n}^2$ tends to zero in $L_1(\Omega)$ as $n \to \infty$, because $u \in L_{2,2}^M$.

Next, it is obvious that $(u^n)^2 \to u^2$ in $L_2([0,T] \times \Omega)$ and since

$$\sum_{k=0}^{n-1} a_{k,n}^2 = \sum_{k=0}^{n-1} \frac{(M_{t_{k+1}} - M_{t_k})^2}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} (u_s^n)^2 ds,$$

using the reasoning similar to that used in proving of the Proposition, we conclude that

$$\sum_{k=0}^{n-1} a_{k,n}^2 \to \int_0^T u_s^2 ds$$

in $L_1(\Omega)$, as $n \to \infty$.

Finally, by virtue of the Cauchy-Bunyakovski inequality, we have

$$\Big| \sum_{k=0}^{n-1} a_{k,n} b_{k,n} \Big| \leq \left(\sum_{k=0}^{n-1} a_{k,n}^2 \right)^{1/2} \left(\sum_{k=0}^{n-1} b_{k,n}^2 \right)^{1/2}$$

and therefore $\sum_{k=0}^{n-1} a_{k,n} b_{k,n}$ tends to zero in $L_1(\Omega)$ as $n \to \infty$.

Summing up the above obtained limit expressions, we complete the proof of theorem.

Definition 3. The stochastic process $U_t(\omega)$ is called an anticipative Poisson semimartingale, if it has the representation

$$U_t(\omega) = U_0(\omega) + \int_0^t v_s(\omega) ds + \int_0^t u_s(\omega) \delta M_s(\omega),$$

where the last integral is the Skorokhod anticipative integral. In this case we use the notation $dU_t = v_t dt + u_t \delta M_t$.

Theorem 2. *If U_t is an anticipative Poisson semimartingales with $dU_t = u_t\delta M_t$, $u \in L^M_{2,2}$ and $F \in C^2_b$, then the process $F(U_t)$ admits the following integral representation*

$$F(U_t) = F(U_0) + \int_0^t F^{'}(U_{s-})u_s\delta M_s + \int_0^t D^M_s[F^{'}(U_{s-})]u_s\delta M_s + \frac{1}{2}\int_0^t F^{''}(U_{s-})u_s^2 ds$$

$$+\int_0^t D^M_s[F^{'}(U_{s-})]u_s ds + \sum_{0<s\leq t}\{F(U_s) - F(U_{s-}) - F^{'}(U_{s-})\Delta U_s\}.$$

Proof. It is obvious that

$$F(U_t) - F(U_0) = \sum_{k=0}^{n-1}[F(U_{t_{k+1}}) - F(U_{t_k})] = \sum_{k=0}^{n-1} F^{'}(U_{t_k})(U_{t_{k+1}} - U_{t_k})$$

$$+\frac{1}{2}\sum_{k=0}^{n-1} F^{''}(\overline{U}_{t_k})(U_{t_{k+1}} - U_{t_k})^2,$$

where $\overline{U}_{t_k}$ is a random intermediate point between U_{t_k} and $U_{t_{k+1}}$.

Due to the Proposition 3.2 [9], we can write

$$\sum_{k=0}^{n-1} F^{'}(U_{t_k})(U_{t_{k+1}} - U_{t_k}) = \sum_{k=0}^{n-1} F^{'}(U_{t_k})\int_{t_k}^{t_{k+1}} u_s\delta M_s$$

$$= \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} F^{'}(U_{t_k})u_s\delta M_s + \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} D_sF^{'}(U_{t_k})u_s ds + \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} D_sF^{'}(U_{t_k})u_s\delta M_s.$$

Using the reasoning similar to that used in proving of the Theorem 1, one can ascertain that the right side of the above expression is tends to

$$\int_0^t F^{'}(U_{s-})u_s\delta M_s + \int_0^t D^M_s[F^{'}(U_{s-})]u_s\delta M_s + \int_0^t D^M_s[F^{'}(U_{s-})]u_s ds$$

in $L_1(\Omega)$, as $n \to \infty$.

On the other hand, using the Proposition and Theorem 1, due to the continuity of $F^{''}$, one can conclude that

$$\frac{1}{2}\sum_{k=0}^{n-1} F^{''}(\overline{U}_{t_k})(U_{t_{k+1}} - U_{t_k})^2 \to$$

$$\to \frac{1}{2}\int_0^t F^{''}(U_{s-})u_s^2 ds + \sum_{0<s\leq t}\{F(U_s) - F(U_{s-}) - F^{'}(U_{s-})\Delta U_s\}$$

in $L_1(\Omega)$, as $n \to \infty$.

Summing up the above obtained limit expressions, we complete the proof of theorem.

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