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A DIFFERENCE SCHEME REPRESENTATION FOR A NONLINEAR KIRCHHOFF EQUATION

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Abstract. The initial boundary value problem for the dynamic string quation $w_{tt} - \big(\lambda + \frac{2}{\pi}\int_0^L w_x^2(x,t)\,dx\big)w_{xx}(x,t) = 0$ is considered. To solve it, the difference scheme is written, which is represented in the form convenient for both solution and investigation since the eigenfunctions of a difference operator are use as a basis.

Keywords and phrases: Kirchhoff string equation, difference scheme.

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1. Statement of the problem. Let us consider the nonlinear equation

$$w_{tt}(x,t) - \left(\lambda + \frac{2}{\pi}\int_0^\pi w_x^2(x,t)\,dx\right)w_{xx}(x,t) = 0, \tag{1}$$
$$0 < x < \pi, \quad 0 < t \le T,$$

with the initial boundary conditions

$$w(x,0) = w^0(x), \quad w_t(x,0) = w^1(x), \tag{2}$$
$$w(0,t) = w(\pi,t) = 0, \tag{3}$$
$$0 \le x \le \pi, \quad 0 \le t \le T.$$

Equation (1) describing the string vibration was obtained by Kirchhoff in 1876 [1]. A great number of works is dedicated to the investigation of this equation and its generalizations (see e.g. [2] and the bibliography therein).

2. Difference scheme. On the intervals $[0,\pi]$ and $[0,T]$ of the change of arguments x and t we introduce the nets $\omega_h = \{x_i = ih,\ i = 0,1,\dots,N\}$ and $\omega_\tau = \{t_m = m\tau,\ m = 0,1,\dots,N\}$, and to the rectangle $[0,\pi]\times[0,T]$ we put into correspondence the net

$$\Omega_{h\tau} = \omega_h \times \omega_\tau = \{(x_i,t_m),\ \ i = 0,1,\dots,N,\ \ m = 0,1,\dots,M\},$$

where h and τ are the steps for which we have $h = \frac{\pi}{N}$, $\tau = \frac{T}{M}$. The value of some function defined on the net $\Omega_{h\tau}$ at the node (x_i,t_m) is denoted by w_i^m. Let us approximate equation (1) and conditions (2), (3) by means of a difference scheme which using the standard notation [3] is written in the form

$$w_{\bar{t}t,i}^m - \frac{1}{2}\left(\lambda + \frac{1}{\pi}\sum_{p=1,-1} h\sum_{j=1}^N (w_{\bar{x},j}^{m+p})^2\right)\sum_{r=1,-1} w_{\bar{x}x,i}^{m+r} = 0, \tag{4}$$
$$i = 1,2,\dots,N-1, \quad m = 1,2,\dots,M-1,$$
$$w_i^0 = \omega_i^0, \quad w_i^1 = \omega_i^1, \quad i = 1,2,\dots,N-1, \tag{5}$$
$$w_0^m = w_N^m = 0, \quad m = 0,1,\dots,M, \tag{6}$$

where $\omega_i^0 = w_i^0$, $\quad \omega_i^1 = w^0(x_i) + \tau w^1(x_i) + \frac{\tau^2}{2}\big(\lambda + \frac{2}{\pi}\int_0^\pi (w^{0\prime}(x))^2 dx\big) w^{0\prime\prime}(x_i)$.

Under the error of the difference scheme (4)-(6) we understand the net function

$$\Delta w_i^m = w(x_i, t_m) - w_i^m. \tag{7}$$
$$i = 0, 1, \ldots, N, \quad m = 0, 1, \ldots, M.$$

A system of equations for the error has the form

$$\Delta w_{tt,i}^m - \frac{1}{2}\left(\lambda + \frac{1}{\pi}\sum_{p=1,-1} h \sum_{j=1}^{N} (w_{\overline{x},j}^{m+p})^2\right) \sum_{r=1,-1} \Delta w_{\overline{x}x,i}^{m+r}$$
$$- \frac{1}{2\pi}\left[\left(\sum_{p=1,-1} h \sum_{j=1}^{N} (w_{\overline{x}}(x_j, t_{m+p}) + w_{\overline{x},j}^{m+p})\right)\Delta w_{\overline{x},j}^{m+p}\right] \sum_{r=1,-1} w_{\overline{x}x.i}^{m+r}$$
$$= \psi_i^{m+1,m-1}, \quad i = 1, 2, \ldots, N-1, \quad m = 1, 2, \ldots, M-1, \tag{8}$$

where

$$\psi_i^{m+1,m-1} = w_{\overline{t}t}(x_i, t_m) - \frac{1}{2}\left(\lambda + \frac{1}{\pi}\sum_{p=1,-1} h \sum_{j=1}^{N} (w_{\overline{x}}(x_j, t_{m+p}))^2\right)$$
$$\times \sum_{r=1,-1} w_{\overline{x}x}(x_i, t_{m+r}).$$

From (5),(6) and (2), (3) we obtain

$$\Delta w_i^0 = 0, \quad \Delta w_i^1 = \frac{\tau^3}{3!} w_{ttt}(x_i, \theta_i), \quad 0 \le \theta_i \le \tau, \quad i = 1, 2, \ldots, N-1, \tag{9}$$
$$\Delta w_0^m = \Delta w_N^m = 0, \quad m = 0, 1, \ldots, M. \tag{10}$$

3. Replacement of the basis. Using the values of the solution of the difference scheme (4)-(6) at the internal nodes of the set ω_h on the m-th layer, i.e. for $t = t_m$, we form the vector $\boldsymbol{w}^m$. Thus $\boldsymbol{w}^m = (w_i^m)_{i=1}^{N-1}$. Let us write the vector $\boldsymbol{w}^m$ in terms of the basis $\{\boldsymbol{e}^i\}_{i=1}^{N-1}$, where the basis vector $\boldsymbol{e}^i$ is the ort $\boldsymbol{e}^i = (\delta_{ij})_{j=1}^{N-1}$, δ_{ij} is the Kronecker symbol. We have

$$\boldsymbol{w}^m = \sum_{i=1}^{N-1} w_i^m \boldsymbol{e}^i. \tag{11}$$

Let us replace the basis. On the net ω_h, we consider the following problem of eigenvalues: find a net function μ_j, $j = 0, 1, \ldots, N$, and a constant λ such that

$$\mu_{\overline{x}x,j} + \lambda\mu_j = 0, \quad j = 1, 2, \ldots, N-1, \tag{12}$$
$$\mu_0 = \mu_N = 0. \tag{13}$$

As it is known, at any rate for sufficiently large N there exist $N-1$ linearly independent solutions of this problem, the i-th solution, $i = 1, 2, \ldots, N-1$, has the form [3] $\mu_j^i =$

$\sqrt{\frac{2h}{\pi}}\sin ijh$, $\lambda_i=\frac{4}{h^2}\sin^2\frac{ih}{2}$, $j=1,2,\dots,N-1$, $\mu_0^i=\mu_N^i=0$, also, $A=(A_{ij})_{i,j=1}^{N-1}$, where $A_{ij}=\mu_j^i$ is an orthonormalized matrix.

Let us consider the basis $\{\boldsymbol{\mu}^i\}_{i=1}^{N-1}$, where the basis vector $\boldsymbol{\mu}^i=(\mu_j^i)_{j=1}^{N-1}$. We write the vector $\boldsymbol{w}^m$ in the new basis

$$\boldsymbol{w}^m=\sum_{i=1}^{N-1}v_i^m\boldsymbol{\mu}^i,\quad m=0,1,\dots,M. \tag{14}$$

Let us complement the set of coefficients of expansion (14) with values $v_0^m=v_N^m=0$ and rewrite the difference scheme (4)-(6) using v_i^m. Taking (11)-(14) into account, we obtain

$$v_{\bar{t}t,i}^m+\frac{1}{2}\lambda_i\left(\lambda+\frac{1}{\pi}\sum_{p=1,-1}h\sum_{j=1}^{N-1}\lambda_j(v_j^{m+p})^2\right)\sum_{r=1,-1}v_i^{m+r}=0, \tag{15}$$

$$i=1,2,\dots,N-1,\quad m=1,2,\dots,M-1,$$

$$v_i^0=\nu_i^0,\quad v_i^1=\nu_i^1,\quad i=1,2,\dots,N-1, \tag{16}$$

$$v_0^m=v_N^m=0,\quad m=0,1,\dots,M. \tag{17}$$

Here $\nu_1^p,\nu_2^p,\dots,\nu_{N-1}^p$ is the solution of the system of linear algebraic equations $\sum\limits_{j=1}^{N-1}A_{ij}\nu_j=\omega_i^p$, $i=1,2,\dots,N-1$, $p=0,1$.

Now, let rewrite the system for error (8)–(10) in the new basis. Using the values Δw_i^m defined by (7) we construct the vector $\Delta\boldsymbol{w}^m=(\Delta w_i^m)_{i=1}^{N-1}$ and write it in the form

$$\Delta\boldsymbol{w}^m=\sum_{i=1}^{N-1}\Delta v_i^m\boldsymbol{\mu}^i,\quad m=0,1,\dots,M. \tag{18}$$

Let us complement the set of coefficients of expansion (18) with values $\Delta v_0^m=\Delta v_N^m=0$. We introduce into consideration the vector $\boldsymbol{w}(x,t_m)=(w(x_i,t_m))_{i=1}^{N-1}$ and the values $v(x_i,t_m)$ from the expansion

$$\boldsymbol{w}(x,t_m)=\sum_{i=1}^{N-1}v(x_i,t_m)\boldsymbol{\mu}^i. \tag{19}$$

Taking into account (12)–(14) and (18), (19), by (8)–(10) we obtain

$$\Delta v_{\bar{t}t,i}^m+\frac{1}{2}\lambda_i\left(\lambda+\frac{1}{\pi}\sum_{p=1,-1}h\sum_{j=1}^{N-1}\lambda_j(v_j^{m+p})^2\right)\sum_{r=1,-1}\Delta v_i^{m+r}$$

$$+\frac{1}{2\pi}\lambda_i\left(\sum_{p=1,-1}h\sum_{j=1}^{N-1}\lambda_j(v(x_j,t_{m+p})+v_j^{m+p})\Delta v_j^{m+p}\right)\sum_{\tau=1,-1}v_i^{m+r}$$

$$=\varphi_i^{m+1,m-1},\quad i=1,2,\dots,N-1,\quad m=1,2,\dots,M-1, \tag{20}$$

$$\Delta v_i^0=0,\quad \Delta v_i^1=\Delta\omega_i,\quad i=1,2,\dots,N-1, \tag{21}$$

$$\Delta v_0^m=\Delta v_N^m=0,\quad m=0,1,\dots,M. \tag{22}$$

Here $\varphi_1^{m+1,m-1}$, $\varphi_2^{m+1,m-1}, \dots, \varphi_{N-1}^{m+1,m-1}$ and $\Delta\omega_1, \Delta\omega_2, \dots, \Delta\omega_{N-1}$ are respectively the solutions of the systems of algebraic equations $\sum\limits_{j=1}^{N-1} A_{ij}\varphi_j = \psi_i^{m+1,m-1}$, $i = 1, 2, \dots, N-1$, and $\sum\limits_{j=1}^{N-1} A_{ij}\Delta\omega_j = \Delta w_i^1$, $i = 1, 2, \dots, N-1$.

Comparing the initial and the transformed systems, we come to a conclusion that for solution of the difference scheme, system (15)–(17) is more convenient than system (4)-(6), whereas in investigating the difference method convergence, it is less difficult to obtain a priori estimates from system (20)–(22) as compared with the case of using system (8)–(10). To conclude, it should be noted that by solving the system layer-by-layer and finding $v_1^m, v_2^m, \dots, v_{N-1}^m$ from (15)-(17), we obtain, by virtue of (14), $w_1^m, w_2^m, \dots, w_{N-1}^m$.

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