Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 25, 2011

THE INTERIOR AND EXTERIOR DIRICHLET BOUNDARY VALUE PROBLEMS OF STATICS OF THERMO-ELECTRO-MAGNETO ELASTICITY THEORY

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Abstract. We investigate the interior and exterior Dirichlet boundary value problems for the system of statics of the thermo-electro-magneto elasticity theory. Using the potential method and the theory of integral equations we prove the existence results. We show that the solutions can be represented by the single and double layer potentials whoes unknown densities are defined by the uniquely solvable integral equations. We study the problems in regular Hölder function spaces.

Keywords and phrases: Thermo-electro-magneto-elasticity, Dirichlet problem, potential method, integral equations.

AMS subject classification: 35J57, 74F05, 74F15, 74B05.

Here we prove the existence theorems for the interior and exterior Dirichlet boundary value problems (BVP) of statics of thermo-electro-magneto elasticity theory. Let Ω^+ be a bounded domain in $\mathbb{R}^3$ with a smooth boundary $S = \partial\Omega^+$ and $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$. Assume that the domains $\overline{\Omega^\pm}$ are filled with an anisotropic homogeneous material with thermo-electro-magneto-elastic properties.

Dirichlet problems $(D)^\pm$: Find a regular solution vector $U = (u, \varphi, \psi, \vartheta)^\top \in [C^1(\overline{\Omega^+})]^6 \bigcap [C^2(\Omega^+)]^6$ (respectively $U \in [C^1(\overline{\Omega^-})]^6 \bigcap [C^2(\Omega^-)]^6$, to the system of equations

$$A(\partial)\, U = 0 \quad \text{in} \quad \Omega^\pm\,, \tag{1}$$

satisfying the Dirichlet type boundary conditions

$$\big\{ U \big\}^\pm = f \quad \text{on} \quad S,$$

where $A(\partial)$ is a nonselfadjoint strongly elliptic matrix partial differential operator generated by the equations of statics of the theory of thermo-electro-magneto-elasticity,

$$A(\partial) := \begin{bmatrix} [c_{rjkl}\,\partial_j\,\partial_l]_{3\times 3} & [e_{lrj}\,\partial_j\partial_l]_{3\times 1} & [q_{lrj}\,\partial_j\partial_l]_{3\times 1} & [-\lambda_{rj}\,\partial_j]_{3\times 1} \\ [-e_{jkl}\,\partial_j\partial_l]_{1\times 3} & \varkappa_{jl}\,\partial_j\,\partial_l & a_{jl}\,\partial_j\,\partial_l & -p_j\,\partial_j \\ [-q_{jkl}\,\partial_j\partial_l]_{1\times 3} & a_{jl}\,\partial_j\,\partial_l & \mu_{jl}\,\partial_j\,\partial_l & -m_j\,\partial_j \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}\,\partial_j\,\partial_l \end{bmatrix}_{6\times 6},$$

the symbols $\{\cdot\}^\pm$ denote the one sided limits (the trace operators) on $\partial\Omega^\pm$ from $\Omega^\pm$, the summation over the repeated indices is meant from 1 to 3; ∂_j denotes partial differentiation with respect to x_j, $\partial_j := \partial/\partial x_j$. The components of the vector $U = (u_1, u_2, u_3, \varphi, \psi, \vartheta)^\top$ have the following physical sense: the first three components correspond to the elastic displacement vector $u = (u_1, u_2, u_3)^\top$, the forth and fifth ones, φ and ψ, are respectively electric and magnetic potentials, and the sixth component ϑ stands for the temperature distribution; c_{rjkl} are the elastic constants, e_{jkl} are the piezoelectric constants, q_{jkl} are the piezomagnetic constants, $\varkappa_{jk}$ are the dielectric (permittivity) constants, μ_{jk} are the magnetic permeability constants, a_{jk} are the coupling coefficients connecting electric and magnetic fields, p_j and m_j are constants characterizing the relation between thermodynamic processes and electromagnetic effects, λ_{jk} are the thermal strain constants, η_{jk} are the heat conductivity coefficients. These constants satisfy specific symmetry conditions ([1], [2], [3]).

In our analysis we need special asymptotic conditions at infinity in the case of unbounded domains.

Definition. We say that a continuous vector $U = (u, \varphi, \psi, \vartheta)^\top \equiv (U_1, \cdots U_6)^\top$ in the domain Ω^- has the property $\mathrm{Z}(\Omega^-)$, if the following conditions are satisfied

$$\widetilde{U}(x) := (u(x), \varphi(x), \psi(x))^\top = \mathcal{O}(1), \qquad U_6(x) = \vartheta(x) = \mathcal{O}(|x|^{-1}) \text{ as } |x| \to \infty,$$

$$\lim_{R\to\infty} \frac{1}{4\pi R^2} \int\limits_{\Sigma_R} U_k(x)\, d\Sigma_R = 0, \quad k = \overline{1,5},$$

where Σ_R is a sphere centered at the origin and radius R.
Denote by $\Gamma(x-y) = [\Gamma_{kj}(x-y)]_{6\times 6}$ the matrix of fundamental solutions of the operator $A(\partial)$,

$$A(\partial)\Gamma(x-y) = I_6\, \delta(x-y),$$

where $\delta(\cdot)$ is the Dirac's delta distribution and I_6 stands for the unit 6×6 matrix. Applying the generalized Fourier transform the fundamental matrix is constructed explicitly and its properties near the origin and at infinity are established. With the help of the fundamental matrix we construct the generalized single and double layer potentials,

$$V(h)(x) = \int\limits_S \Gamma(x-y)\, h(y)\, dS_y, \quad x \in \mathbb{R}^3 \setminus S,$$

$$W(h)(x) = \int\limits_S [\mathcal{P}(\partial_y, n(y))\Gamma^\top(x-y)]^\top\, h(y)\, dS_y, \quad x \in \mathbb{R}^3 \setminus S,$$

where $h = (h_1, \cdots, h_6)^\top$ is a density vector and the boundary operator $\mathcal{P}(\partial, n)$ is given by the formula

$$\mathcal{P}(\partial, n) = \big[\ \mathcal{P}_{pq}(\partial, n)\ \big]_{6\times 6}$$

$$= \begin{bmatrix} [c_{rjkl}\, n_j\, \partial_l]_{3\times 3} & [-e_{lrj}\, n_j\, \partial_l]_{3\times 1} & [-q_{lrj}\, n_j\, \partial_l]_{3\times 1} & [0]_{3\times 1} \\ [e_{jkl}\, n_j\, \partial_l]_{1\times 3} & \varkappa_{jl}\, n_j\, \partial_l & a_{jl}\, n_j\, \partial_l & 0 \\ [q_{jkl}\, n_j\, \partial_l]_{1\times 3} & a_{jl}\, n_j\, \partial_l & \mu_{jl}\, n_j\, \partial_l & 0 \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}\, n_j\, \partial_l \end{bmatrix}_{6\times 6},$$

$n = (n_1, n_2, n_3)$ is the outward unit normal vector with respect to Ω^+ at the point $x \in \partial\Omega^+$.

The qualitative and mapping properties of the layer potentials are described by the following theorems.

Theorem 1. *The generalized single and double layer potentials solve the homogeneous differential equation* $A(\partial)U = 0$ *in* $\mathbb{R}^3 \setminus S$ *and possess the property* $\mathrm{Z}(\Omega^-)$.

Theorem 2. *Let* $S = \partial\Omega^{\pm} \in C^{m,\kappa}$ *with integers* $m \geq 1$ *and* $k \leq m-1$, *and* $0 < \kappa' < \kappa \leq 1$. *Then the operators*

$$V \ : \ [C^{k,\,\kappa'}(S)]^6 \to [C^{k+1,\,\kappa'}(\overline{\Omega^{\pm}})]^6,$$
$$W \ : \ [C^{k,\,\kappa'}(S)]^6 \to [C^{k,\,\kappa'}(\overline{\Omega^{\pm}})]^6,$$

are continuous.

For any $g \in [C^{0,\kappa'}(S)]^6$, $h \in [C^{1,\kappa'}(S)]^6$, *and any* $x \in S$ *we have the following jump relations:*

$$\{\, V(g)(x)\,\}^{\pm} = V(g)(x) = \mathcal{H}\, g(x), \tag{2}$$

$$\{\, \mathcal{T}(\partial_x, n(x))\, V(g)(x)\,\}^{\pm} = [\, \mp 2^{-1} I_6 + \mathcal{K}\,]\, g(x), \tag{3}$$

$$\{\, W(g)(x)\,\}^{\pm} = [\, \pm 2^{-1} I_6 + \mathcal{N}\,]\, g(x), \tag{4}$$

$$\{\, \mathcal{T}(\partial_x, n(x))\, W(h)(x)\,\}^{+} = \{\, \mathcal{T}(\partial_x, n(x))\, W(h)(x)\,\}^{-} = \mathcal{L}\, h(x), \quad m \geq 2, \tag{5}$$

where

$$\mathcal{H}\, g(x) := \int_S \Gamma(x-y)\, g(y)\, dS_y\,, \tag{6}$$

$$\mathcal{K}\, g(x) := \int_S \mathcal{T}(\partial_x, n(x))\, \Gamma(x-y)\ \ g(y)\, dS_y\,, \tag{7}$$

$$\mathcal{N}\, g(x) := \int_S [\, \mathcal{P}(\partial_y, n(y))\, \Gamma^{\top}(x-y)\,]^{\top}\, g(y)\, dS_y\,, \tag{8}$$

$$\mathcal{L}\, h(x) := \lim_{\Omega^{\pm}\ni z\to x\in S} \mathcal{T}(\partial_z, n(x)) \int_S [\, \mathcal{P}(\partial_y, n(y))\, \Gamma^{\top}(z-y)\,]^{\top}\, h(y)\, dS_y\,, \tag{9}$$

and the matrix boundary operator $\mathcal{T}(\partial, n)$, *called the generalized stress operator, reads as*

$$\mathcal{T}(\partial, n) := \begin{bmatrix} [c_{rjkl}\, n_j\, \partial_l]_{3\times3} & [e_{lrj}\, n_j\, \partial_l]_{3\times1} & [q_{lrj}\, n_j\, \partial_l]_{3\times1} & [-\lambda_{rj}\, n_j]_{3\times1} \\ [-e_{jkl}\, n_j\, \partial_l]_{1\times3} & \varkappa_{jl}\, n_j\, \partial_l & a_{jl}\, n_j\, \partial_l & -p_j\, n_j \\ [-q_{jkl}\, n_j\, \partial_l]_{1\times3} & a_{jl}\, n_j\, \partial_l & \mu_{jl}\, n_j\, \partial_l & -m_j\, n_j \\ [0]_{1\times3} & 0 & 0 & \eta_{jl}\, n_j\, \partial_l \end{bmatrix}_{6\times6}.$$

Applying the jump relations (2)-(5) and with the help of Fredholm properties of the integral operators (6)-(9), also using the uniqueness theorems for the above formulated problems presented in [4], we prove the following existence results for the interior and exterior Dirichlet boundary value problems of statics of thermo-electro-magneto-elasticity.

Theorem 3. *Let $S \in C^{2,\kappa}$ and $f \in [C^{1,\kappa'}(S)]^6$, where $0 < \kappa' < \kappa \leq 1$. Then the interior Dirichlet BVP is uniquely solvable in the space of regular vector-functions $[C^{1,\kappa'}(\overline{\Omega^+})]^6 \bigcap [C^2(\Omega^+)]^6$ for arbitrary f and the solution is represented by the double layer potential, $U = W(h)$, where the density vector function $h \in [C^{1,\kappa'}(S)]^6$ is defined by the uniquely solvable singular integral equation*

$$[\,2^{-1}\,I_6 + \mathcal{N}\,]\,h = f \quad on \quad S.$$

Theorem 4. *Let $S \in C^{2,\kappa}$ and $f \in [C^{1,\kappa'}(S)]^6$, where $0 < \kappa' < \kappa \leq 1$. Then the exterior Dirichlet boundary value problem is uniquely solvable in the space of regular vector-functions $[C^{1,\kappa'}(\overline{\Omega^-})]^6 \bigcap [C^2(\Omega^-)]^6 \bigcap \mathrm{Z}(\Omega^-)$, and the solution is representable in the form of linear combination of the single and double layer potentials*

$$U(x) = W(h)(x) + \alpha\, V(h)(x),$$

where α is an arbitrary positive constant and the density vector function $h = (h_1, ..., h_6)^\top \in [C^{1,\kappa'}(S)]^6$ is defined by the uniquely solvable singular integral equation

$$[\,-2^{-1}\,I_6 + \mathcal{N} + \alpha\,\mathcal{H}\,]\,h = f \quad on \quad S.$$

Theorem 5. *Let $S \in C^{2,\kappa}$ and $f \in [C^{1,\kappa'}(S)]^6$, where $0 < \kappa' < \kappa \leq 1$. Then the interior and exterior Dirichlet boundary value problems are uniquely solvable in the spaces of regular vector-functions $[C^{1,\kappa'}(\overline{\Omega^+})]^6 \bigcap [C^2(\Omega^+)]^6$ and $[C^{1,\kappa'}(\overline{\Omega^-})]^6 \bigcap [C^2(\Omega^-)]^6 \bigcap \mathrm{Z}(\Omega^-)$, respectively, and the solution is representable in the form of the single layer potential, $U = V(h)$, where the density vector function $h \in [C^{0,\kappa'}(S)]^6$ is defined by the uniquely solvable integral equation*

$$\mathcal{H}\,h = f \quad on \quad S.$$

From Theorem 5 the following assertion follows (see [3], [5]).

Corollary. Let $S \in C^{2,\kappa}$ and $U \in [C^{1,\kappa'}(\overline{\Omega^\pm})]^6 \bigcap [C^2(\Omega^\pm)]^6$ with $0 < \kappa' < \kappa \leq 1$ be an arbitrary solution of the homogeneous equation (1), possessing the property $\mathrm{Z}(\Omega^-)$ in the case of the exterior domain Ω^-. Then U is uniquely representable in the form of the single layer potential

$$U(x) = V\left(\mathcal{H}^{-1}\{U\}^\pm\right)(x)\,, \quad x \in \Omega^\pm,$$

where $\mathcal{H}^{-1}$ is the inverse to the operator

$$\mathcal{H} : [C^{0,\kappa'}(S)]^6 \to [C^{1,\kappa'}(S)]^6\,.$$

Theorem 5 and Corollary play a crucial role in the study of mixed BVPs.

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Received 05.05.2011; accepted 11.10.2011.

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