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SOME NEW RESULTS ON VALIDITY CONDITIONS FOR GAUSS HYPERGEOMETRICAL FUNCTIONS RELATIONS

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Abstract. Expansion of validity conditions of the known relations for the Gauss hypergeometric function is presented.

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The Gauss hypergeometric (HG) function - the solution of ordinary second order differential equation - plays important (sometimes conclusive) role in many problems of mathematics and of theoretical physics due to it is the base for many special functions of mathematical physics. That is a reason of a great deal of attention which is devoted to researches of HG function (see e.g. [1]-[3] and sources cited there). One of important problem is an expansion of the conditions for the Gauss hypergeometric function well known relations. One of such expansion is presented bellow.

Theorem 1. *If $a = c + n$, $b = c + k$, the formulas of analytic continuity:*

$$F(a, b; c; \xi) \equiv F\left(\begin{matrix} a \\ c \end{matrix}; b; \xi\right)$$

$$= (-1)^{b-c} \frac{\Gamma(c)\,\Gamma(1+a-c)}{\Gamma(b)\,\Gamma(1+a-b)} (-\xi)^{-a} F\left(\begin{matrix} a \\ 1+a-b \end{matrix}; 1+a-c; \xi^{-1}\right), \quad (1a)$$

$$a - b = m,$$

$$= (-1)^{a-c} \frac{\Gamma(c)\,\Gamma(1+b-c)}{\Gamma(a)\,\Gamma(1+b-a)} (-\xi)^{-b} F\left(\begin{matrix} b \\ 1+b-a \end{matrix}; 1+b-c; \xi^{-1}\right), \quad (1b)$$

$$a - b = m,$$

$$c \neq 0, -1, -2, ..., \quad |\arg(-\xi)| < \pi, \quad |\arg(1-\xi)| < \pi, \quad m = 0, 1, ...,$$

remains valid.

Proof. Using the well known relation (see e. g. [1], Eq. 2.9.(2))

$$F\left(\begin{matrix} a \\ c \end{matrix}; b; \xi\right) = (1-\xi)^{c-a-b} F\left(\begin{matrix} c-a \\ c \end{matrix}; c-b; \xi\right), \quad |\arg(1-\xi)| < \pi, \quad (2)$$

which in our case gets the form

$$F\left(\begin{matrix} c+n \\ c \end{matrix}; c+k; \xi\right) = (1-\xi)^{-k-n-c} F\left(\begin{matrix} -n \\ c \end{matrix}; -k; \xi\right), \quad (3)$$

and expanding the function in a series (8), one obtains the finite sum:

$$F\left(\begin{matrix} -n \\ c \end{matrix}; -k; \xi\right) = \sum_{j=0}^{\infty} \frac{(-n)_j (-k)_j}{(c)_j} \frac{\xi^j}{j!} = \sum_{j=0}^{\min(n,k)} \frac{(-n)_j (-k)_j}{(c)_j} \frac{\xi^j}{j!}.$$

Using here the well known relations

$$(k)_j = \Gamma(k+j)/\Gamma(k), \quad (-k)_j = (-1)^j \Gamma(k+1)/\Gamma(k-j+1), \tag{4}$$

let us transform the coefficient in the j-th summand:

$$F\left(\begin{array}{c} -n \\ c \end{array}; -k; \xi\right) = \sum_{j=0}^{k} \frac{\Gamma(c)}{\Gamma(c+j)} \frac{\Gamma(n+1)}{\Gamma(n-j+1)} \frac{\Gamma(k+1)}{\Gamma(k-j+1)} \frac{\xi^j}{j!}, \tag{5}$$

and then let us replace the index of the summation in the (5): $j \to k-j$. After simplifications we get:

$$\begin{aligned} F\left(\begin{array}{c} -n \\ c \end{array}; -k; \xi\right) &= \frac{(-\xi)^k}{(-n)_k (c)_k} \sum_{j=0}^{k} \frac{(1-c-k)_j (-k)_j}{(n-k+1)_j} \frac{\xi^{-j}}{j!} \\ &= \frac{\Gamma(n+1)\Gamma(c)}{\Gamma(n-k+1)\Gamma(k+c)} \xi^k F\left(\begin{array}{c} 1-c-k \\ n-k+1 \end{array}; -k; \xi^{-1}\right), \quad n \geq k \end{aligned} \tag{6}$$

(the relation $\Gamma(\xi)\Gamma(1-\xi) = \pi/\sin(\pi\xi)$ and (4) are used here)). Inserting (6) into (3) we obtain:

$$F\left(\begin{array}{c} a \\ c \end{array}; b; \xi\right) = \frac{\Gamma(a-c+1)\Gamma(c)}{\Gamma(a-b+1)\Gamma(b)} \xi^{b-c} (1-\xi)^{c-a-b} F\left(\begin{array}{c} 1-b \\ 1+a-b \end{array}; c-b; \xi^{-1}\right), \tag{7}$$
$$a = c+n, \quad b = c+k, \quad n \geq k.$$

Using in (7) the formula (2) we get (1a). The expression (1b) follows from this due to the well known symmetry of the HG function.

As it is known, an analytic continuation of the HG function (see e.g. [1], Eq. 2.10.(2))

$$\begin{aligned} F\left(\begin{array}{c} a \\ c \end{array}; b; \xi\right) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-\xi)^{-a} F\left(\begin{array}{c} a \\ 1+a-b \end{array}; 1+a-c; \xi^{-1}\right) \\ &+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-\xi)^{-b} F\left(\begin{array}{c} b \\ 1+b-a \end{array}; 1+b-c; \xi^{-1}\right), \\ &|\arg(-\xi)| < \pi, \quad |\arg(1-\xi)| < \pi, \end{aligned} \tag{8}$$

is valid if the restriction $|a-b| \neq m = 0, 1, ...$ takes place. Now let us prove the next

Theorem 2. *The analytic continuity (8) of the HG function is valid even though* $a = c+n, \ b = c+k, \ n, k \in N$ *and it reduces to the relations (1) in that case.*

Proof. Let us assume in (8) that $a = c+n+\varepsilon, \ b = c+k, \ n \geq k$. Then we will have:

$$
\begin{aligned}
F\left(\begin{matrix} c+n+\varepsilon \\ c \end{matrix}; c+k; \xi\right)
&= \frac{\Gamma(c)\,\Gamma(k-n-\varepsilon)}{\Gamma(c+k)\,\Gamma(-n-\varepsilon)}(-\xi)^{-(c+n+\varepsilon)} \\
&\qquad \times F\left(\begin{matrix} c+n+\varepsilon \\ 1+n-k+\varepsilon \end{matrix}; 1+n+\varepsilon; \xi^{-1}\right) \\
&= (-1)^k \frac{\Gamma(c)\,\Gamma(1+n+\varepsilon)}{\Gamma(c+k)\,\Gamma(1+n-k+\varepsilon)}(-\xi)^{-(c+n+\varepsilon)} \\
&\qquad \times F\left(\begin{matrix} c+n+\varepsilon \\ 1+n-k+\varepsilon \end{matrix}; 1+n; \xi^{-1}\right), \qquad (9) \\
&\qquad n-k=m=0,1,...
\end{aligned}
$$

As far as the Gauss HG function is a continuous function of its parameters, the limit of the expression (9) when $\varepsilon \to 0$ gives

$$
\begin{aligned}
F\left(\begin{matrix} c+n \\ c \end{matrix}; c+k; \xi\right) &= \frac{(-1)^k\Gamma(c)\,\Gamma(1+n)}{\Gamma(c+k)\,\Gamma(1+n-k)} \\
&\quad \times (-\xi)^{-(c+n)} F\left(\begin{matrix} c+n \\ 1+n-k \end{matrix}; 1+n; \xi^{-1}\right),\ n-k=m=0,1,.... \qquad (10)
\end{aligned}
$$

Obviously, (10) reduces to (1a). Similarly, when $a=c+n,\ \ b=c+k,\ \ n \le k$, (8) reduces to (1b).

Analogically it may be proved

Theorem $2'$**.** *The formula of analytic continuity* (*see e.g.* [1], Eq. 2.10.(3))

$$
\begin{aligned}
F\left(\begin{matrix} a \\ c \end{matrix}; b; \xi\right) &= \frac{\Gamma(c)\,\Gamma(b-a)}{\Gamma(b)\,\Gamma(c-a)}(1-\xi)^{-a} F\left(\begin{matrix} a \\ 1+a-b \end{matrix}; c-b; (1-\xi)^{-1}\right) \\
&\quad + \frac{\Gamma(c)\,\Gamma(a-b)}{\Gamma(a)\,\Gamma(c-b)}(1-\xi)^{-b} F\left(\begin{matrix} b \\ 1+b-a \end{matrix}; c-a; (1-\xi)^{-1}\right), \\
&\qquad |\arg(-\xi)| < \pi, \quad |\arg(1-\xi)| < \pi
\end{aligned}
$$

remains valid even though $a=c+n,\ \ b=c+k$, *and reduces to the relations:*

$$
\begin{aligned}
F\left(\begin{matrix} a \\ c \end{matrix}; b; \xi\right) &= (-1)^{b-c}\frac{\Gamma(c)\,\Gamma(1+a-c)}{\Gamma(b)\,\Gamma(1+a-b)}(1-\xi)^{-a} \\
&\qquad \times F\left(\begin{matrix} a \\ 1+a-b \end{matrix}; c-b; (1-\xi)^{-1}\right), \quad a-b=m, \qquad (11a) \\
&= (-1)^{a-c}\frac{\Gamma(c)\,\Gamma(1+b-c)}{\Gamma(a)\,\Gamma(1+b-a)}(1-\xi)^{-b} \\
&\qquad \times F\left(\begin{matrix} b \\ 1+b-a \end{matrix}; c-a; (1-\xi)^{-1}\right), \quad b-a=m, \qquad (11b) \\
&|\arg(-\xi)\,| < \pi, \quad |\arg(1-\xi)\,| < \pi, \qquad m=0,\ 1,\ \ldots
\end{aligned}
$$

Corollary. For the regularized HG function defined by the relation $F_R(a,b;c;\xi) = F(a,b;c;\xi)/\Gamma(c)$ (see e.g. [1], Eq. 2.8.(19)) from the formulas (1) and (11) we get for $a = c+n,\ \ b = c+k$:

$$
\begin{aligned}
F_R\left(\begin{matrix} a \\ c \end{matrix};b;\xi\right) &= (-1)^{b-c}\frac{\Gamma(1+a-c)}{\Gamma(b)}(-\xi)^{-a} \\
&\qquad\times F_R\left(\begin{matrix} a \\ 1+a-b \end{matrix};1+a-c;\xi^{-1}\right), \quad n \geq k; \\
&= (-1)^{a-c}\frac{\Gamma(1+b-c)}{\Gamma(a)}(-\xi)^{-b} \\
&\qquad\times F_R\left(\begin{matrix} b \\ 1+b-a \end{matrix};1+b-c;\xi^{-1}\right), \quad n \leq k;
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^{b-c}\frac{\Gamma(1+a-c)}{\Gamma(b)}(1-\xi)^{-a} \\
&\qquad\times F_R\left(\begin{matrix} a \\ 1+a-b \end{matrix};c-b;(1-\xi)^{-1}\right), \quad n \geq k; \\
&= (-1)^{a-c}\frac{\Gamma(1+b-c)}{\Gamma(a)}(1-\xi)^{-b} \\
&\qquad\times F_R\left(\begin{matrix} b \\ 1+b-a \end{matrix};c-a;(1-\xi)^{-1}\right), \quad n \leq k.
\end{aligned}
$$

Obviously, the restrictions $c \neq 0,-1,...$ are removed for the regularized functions.

Thus we have shown that some well known relations for the HG function remain valid even if some restrictions mentioned in literature on the parameters of the HG function are avoided.

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