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ON VARIATIONAL FORMULATION OF SOME NONLOCAL BOUNDARY VALUE PROBLEM

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Abstract. The nonlocal boundary condition, which is the elementary generalization of a homogeneous Dirichlet boundary condition is considered. For the corresponding boundary problem the parametric quadratic functional is constructed. The necessary and sufficient conditions of the fact that the minimizing function of this functional is the solution of stated problem is given.

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Let us $G =]-a,\, 0[\times[0,\, b[$, $\Gamma_t = \{(t,y)|\ y \in [0,\, b]\}$, $-a \le t \le 0$ and $g \in C^{(1)}[0,\, b]$ for $\forall y \in [0,\, b]$ is a given function. $V \in W_2^1(G)$ is such function that its trace on Γ_{-a} is equal to zero, while on trace on Γ_0 is proportional to $g(y)$ (almost everywhere).

We consider following problem: Find the function $u \in C^{(2)}(G) \cap C(\overline{G})$ satisfying Poisson's equation with classical boundary and one kind of nonlocal integral conditions:

$$V = \{v|\ v \in W_2^1(G), \quad v(-a,y)=0, \quad v(0,y)=\alpha_v g(y), \quad \alpha_v \in R, \quad y \in [0,b]\}.$$

The scalar product in V is reduced from space $W_2^1(G)$:

$$(v_1,\, v_2)_V = \int_{-a}^{0}\int_{0}^{b} \left(v_1(x,y)v_2(x,y) + \frac{\partial v_1(x,y)}{\partial x}\frac{\partial v_2(x,y)}{\partial x} + \frac{\partial v_1(x,y)}{\partial y}\frac{\partial v_2(x,y)}{\partial y} \right) dxdy. \tag{1}$$

Assume, that E is linear, continuous functional defined on Hilbert space V. Then the space of functions

$$V_E = \{v|\ v \in V, \quad v(0,y) = Evg(y)\}$$

is subspace of the space V. Note that for all such functionals, by the Ritz theorem, correspond one and only one $e \in V$, such that $Ev = (e,\, v)_V$, $\forall v \in V$.

So,

$$V_E = \{v|\ v \in V, \quad v(0,y) = (e,\, v)_V g(y)\}. \tag{2}$$

Sometimes we will use notation V_e instead of V_E (for example, $V_\omega = \{v|\ v \in V$, $v(0,y) = (\omega,\, v)_V g(y)\}$).

Equality

$$v(0,y) = (e,\, v)_V g(y) \tag{3}$$

in definition of V_E gives different boundary conditions on Γ_0. In particular, if we take $e = 0$, then (3) gives homogenous Dirichlet boundary condition and therefore it is included in the family of boundary conditions of (3) type.

We consider boundary problem with (3) nonlocal boundary condition.

For a given number $\lambda \in R_+$ and for functions $f \in C(\overline{G})$ and $e \in C^{(1)}(\overline{G})$ let us find function $u \in C^{(2)}(G) \cap C(\overline{G})$ such that:

$$-\Delta u(x,y) + \lambda u(x,y) = f(x,y), \quad (x,y) \in G, \tag{4}$$

$$u(-a,y) = 0, \quad y \in [0,b], \quad \frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial y}(x,b) = 0, \quad x \in]-a,\, 0[, \tag{5}$$

$$u(0,y) = (e,\, u)_V g(y). \tag{6}$$

Note that for the certainly selected function e, condition (6) represents identity on V. It is clear that for such function e, problem (4)-(6) has infinity set of solutions.

Many scientists have been investigating nonlocal boundary value problems for ordinary differential equations and partial differential elliptic equations (see, for example, [1]-[19] and references therein).

Let us E be a linear, bounded functional defined on V: $E : v \to \alpha_v$. Certainly, for such functional E we have $V_E = V$.

Let us denote by $D(\overline{G})$ the lineal of all the real functions $\overline{v} = (v(x,y), v(0,y))$ satisfying the following conditions:

1. $\overline{v}$ is defined almost everywhere on $\overline{G}$, and the boundary value $v(0,y)$ (the value on the boundary Γ_0) is proportional to $g(y)$ almost everywhere, i.e. for all $\overline{v} \in D(\overline{G})$ corresponds $\alpha_{\overline{v}}$ such that $v(0,y) = \alpha_{\overline{v}} g(y)$ almost everywhere on Γ_0;

2. $v \in L_2(G)$.

Two functions $\overline{v}_1$ and $\overline{v}_2$ are assumed as the same element of $D(\overline{G})$ if $v_1(x,y) = v_2(x,y)$ almost everywhere on $\overline{G} \backslash \Gamma_0$ and $\alpha_{\overline{v}_1} = \alpha_{\overline{v}_2}$.

Let us $\overline{Q}$ is the closer of the rectangle $Q = \{(x,y) |\ 0 < x < a,\ \ 0 < y < b\}$ and define on $D(\overline{G})$ the operator of symmetrical extension τ as follows:

$$(\tau v)(x,y) = \begin{cases} v(x,y), & (x,y) \in \overline{G}, \\ -v(-x,y) + 2v(0,y), & (x,y) \in \overline{Q}. \end{cases}$$

For two arbitrary functions $\overline{v}_1$ and $\overline{v}_2$ from the lineal $D(\overline{G})$ we define the scalar product

$$[\overline{v}_1,\ \overline{v}_2] = \int\limits_{-a}^{a} \int\limits_{-a}^{x} \int\limits_{0}^{b} \widetilde{v}_1(s,y) \widetilde{v}_2(s,y) ds dx dy,$$

where $\widetilde{v}_i(s,y) = (\tau \overline{v}_i)(s,y), \quad i = 1,2.$

After the introduction of the scalar product the lineal $D(\overline{G})$ becomes the pre-Hilbert space, which we denote by $H(\overline{G})$. The norm originated from this scalar product we denote by $\|\cdot\|_H$:

$$\|v\|_H^2 = \int\limits_{-a}^{a} \int\limits_{-a}^{x} \int\limits_{0}^{b} \widetilde{v}^2(s,y) ds dx dy.$$

Theorem 1. *The norm defined on the* $H(\overline{G})$ *by the formula*

$$\|v\|^2 = \|v\|^2_{L_2(G)} + \alpha_v^2$$

is equivalent to the norm $\|\cdot\|_H$.

Consequence. $H(\overline{G})$ *is the Hilbert space.*

Let, $e \in V$ is fixed element and area of definition of the operator $A = -\Delta + \lambda I$ is the lineal $D_A(\overline{G})$ of the functions v defined on $\overline{G}$, all functions $v \in V$ of which satisfies the following conditions:

$$1.\ v \in C^{(\infty)}(\overline{G}), \quad \frac{\partial^k v}{\partial x^k}(0, y) = 0, \quad \forall y \in [0, b],$$

$$\frac{\partial^k v}{\partial y^k}(x, 0) = \frac{\partial^k v}{\partial y^k}(x, b) = 0, \ \ \forall x \in [-a, 0], \ \ k = 1,\ 2,\ ...;$$

$$2.\ v(-a, y) = 0, \quad (e,\, v)_V \alpha_v.$$

Theorem 2. *The lineal* $D_A(\overline{G})$ *is dense in the space* $H(\overline{G})$.

Theorem 3. *There exists number* $\gamma > 0$ *such that if*

$$(e,\, e)_V < \gamma, \tag{7}$$

when A *is positively defined on the lineal* $D_A(\overline{G})$.

So, if the condition (7) is fulfilled then A is positive definite operator defined on the lineal $D_A(\overline{G})$ which is dense in the space $H(\overline{G})$ and for problem (1)-(3) we can use the standard way of the variational formulation [20]. Let us introduce the new scalar product on $D_A(\overline{G})$

$$\begin{gathered}[v_1, v_2]_A = [Av_1, v_2] = \int\limits_{-a}^{a}\int\limits_{-a}^{x}\int\limits_{0}^{b}\left(\frac{\partial\widetilde{v}_1(s,y)}{\partial s}\frac{\partial\widetilde{v}_2(s,y)}{\partial s}\right. \\ \left.+\frac{\partial\widetilde{v}_1(s,y)}{\partial y}\frac{\partial\widetilde{v}_2(s,y)}{\partial y} + \lambda\widetilde{v}_1(s,y)\widetilde{v}_2(s,y)\right) dsdxdy - 2\alpha_{v_1}\alpha_{v_2}\int\limits_0^b g^2(y)dy.\end{gathered} \tag{8}$$

For corresponding norm we use the notation $\|\cdot\|_A$.

After introducing the scalar product (8) the lineal $D_A(\overline{G})$ becomes the pre-Hilbert space which we denote by $S_A(\overline{G})$. By $H_A(\overline{G})$ we denote the Hilbert space obtained after completion of $S_A(\overline{G})$ by the norm $\|\cdot\|_A$.

Theorem 4. *The* $\|\cdot\|_A$ *and* $\|\cdot\|_{W_2^1(G)}$ *defined in the space* $S_A(\overline{G})$ *are equivalent norms.*

So, spaces $H_A(\overline{G})$ and V_e arise from the same functions. The difference is in scalar products which birth equivalent norms.

For every function $\overline{f} = (f, pg) \in H(\overline{G})$ the quadratic functional

$$Fv = [v, v]_A - 2\left[\bar{f}, v\right] \tag{9}$$

has the unique function $u \in H_A(\overline{G})$, which minimizes the functional (9) and satisfies the identity

$$[u,v]_A = \left[\bar{f}, v\right] \tag{10}$$

for every $v \in H_A(\overline{G})$.

The functional (9) in the extended form can be written as

$$\begin{aligned} Fv = 2a \int_{-a}^{0}\int_{0}^{b} \Bigg[\left(\frac{\partial v(x,y)}{\partial x}\right)^2 + \left(\frac{\partial v(x,y)}{\partial y}\right)^2 \\ +\lambda v^2(x,y) - 2f(x,y)v(x,y)\Bigg]\, dxdy \\ + \left(2a^2 \int_0^b \left(g'(y)\right)^2 dy + 2(\lambda a^2 - 1)\int_0^b g^2(y)dy\right)(e,\, v)_V^2 \\ -4\int_{-a}^{0}\int_{0}^{b}(x+a)\left(g'(y)\frac{\partial v(x,y)}{\partial y} + \lambda g(y)v(x,y)\right) dxdy(e,\, v)_V \\ +4\int_{-a}^{0}\int_{0}^{b} g(y)(x+a)((e,\, v)_V f(x,y) \\ +pv(x,y))dxdy - 4pa^2\int_0^b g^2(y)dy(e,\, v)_V. \end{aligned} \tag{11}$$

For (10) we have

$$\begin{aligned} 2a\int_{-a}^{0}\int_{0}^{b}\left(\frac{\partial u(x,y)}{\partial x}\frac{\partial v(x,y)}{\partial x} + \frac{\partial u(x,y)}{\partial y}\frac{\partial v(x,y)}{\partial y} + \lambda u(x,y)v(x,y)\right) dxdy \\ -2\int_{-a}^{0}\int_{0}^{b}(g'(y)(x+a)\left((e,\, u)_V\frac{\partial v(x,y)}{\partial y} + (e,\, v)_V\frac{\partial u(x,y)}{\partial y}\right) dxdy \\ -2\lambda\int_{-a}^{0}\int_{0}^{b}(g(y)(x+a)\left((e,\, u)_V v(x,y) + (e,\, v)_V u(x,y)\right) dxdy \\ +2a^2(e,\, u)_V(e,\, v)_V\int_0^b \left(g'(y)\right)^2 dy + 2(a^2\lambda - 1)(e,\, u)_V(e,\, v)_V\int_0^b g^2(y)dy \\ = 2a\int_{-a}^{0}\int_{0}^{b} f(x,y)v(x,y)dxdy - 2\int_{-a}^{0}\int_{0}^{b} g(y)(x+a)\left((e,\, v)_V f(x,y)\right. \\ \left.+pv(x,y)\right) dxdy + 2p(e,\, v)_V a^2\int_0^b g^2(y)dy. \end{aligned}$$

Theorem 5. *The condition*

$$-u''_{yy}(0,y) + \lambda u(0,y) = f(0,y),$$

or

$$(e,\, u)_V \left(-g''(0,y) + \lambda g(y)\right) = pg(y) \tag{12}$$

is necessary and sufficient in order to the solution u of problem (4)-(6) minimizes the functional (11).

Let us note that if we take $g(y)$ as a some solution of homogeneous equation $-g''(0,y) + \lambda g(y) = 0$, then for $p = 0$ condition (12) will be satisfied and minimizing function of (11) u on $H_A(\overline{G})$ give solution of the problem (4)-(6).

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