Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 25, 2011

STATEMENT AND SOLUTION OF SOME NONLCLASSICAL TWO-DIMENSIONAL PROBLEMS OF THERMOELASTICITY

Khomasuridze N., Janjgava R., Zirakashvili N.

Abstract. In the present paper, we consider the following nonclassical two-dimensional problems of thermoelasticity for homogeneous isotropic bodies. The boundary symmetry or antisymmetry conditions are given on two opposite sides of the rectangular domain; the other two sides of the rectangle are free from stresses and on one of them a temperature disturbance function is given. The problem consists in giving a temperature on the other stress-free side of the rectangle so that a certain linear combination of normal displacements on two segments inside the body which are parallel to this stress-free side would take a prescribed value. The stated problem is solved analytically, using the method of separation of variables.

Keywords and phrases: Non-classical thermo-elasticity problems, method of separation of variables, analytical solutions.

AMS subject classification: 74B05, 74F05.

1. Introduction. In elasticity theory there exist quite a number of problems which could be called nonclassical in view of the fact that the boundary conditions on a part of the boundary surface are either over- or under determined [1], [2] or the conditions on the boundary are related to the conditions inside the body (the so-called nonlocal problems) [3], [4], [5].

In the present paper, we consider the following nonclassical two-dimensional problems of thermoelasticity for homogeneous isotropic bodies.

The boundary symmetry or antisymmetry conditions are given on two opposite sides of the rectangular domain [6]; the other two sides of the rectangle are free from stresses and on one of them a temperature disturbance function is given. The problem consists in giving a temperature on the other stress-free side of the rectangle so that a certain linear combination of normal displacements on two segments inside the body which are parallel to this stress-free side would take a prescribed value.

The stated problem is solved analytically, using the method of separation of variables.

2. Statement of the problems. We consider the plane thermoelastic equilibrium of an isotropic homogeneous body whose cross-section, in the Cartesian system of x, y-coordinates, occupies the domain $\omega = \{0 < x < x_1,\ \ 0 < y < y_1\}$ (Fig. 1).

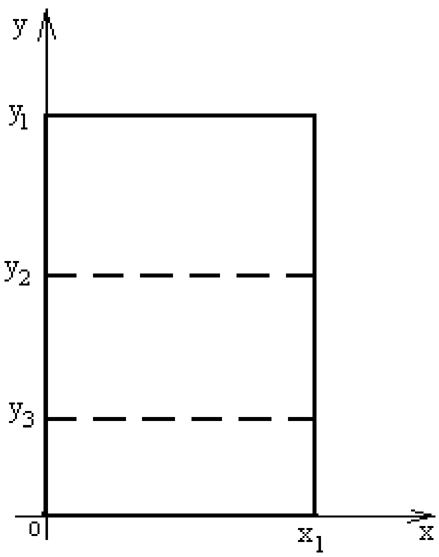

Fig.1.

On the lateral sides of the rectangle ω the following boundary conditions are given [7]:

$$\left.\begin{array}{ll}\text{for } x=x_\alpha: & a)\ T=0,\ \ \sigma_{xx}=0,\ \ v=0 \quad \text{or} \\ & b)\ \partial_x T=0,\ \ u=0,\ \ \sigma_{xy}=0.\end{array}\right\} \qquad (1)$$

The boundary conditions

$$y=0 \ \text{ and } \ y=y_1: \quad \sigma_{yy}=0, \quad \sigma_{yx}=0 \qquad (2)$$

are given on the upper and lower sides of the domain ω , and on the side $y=0$ we have

$$\left.\begin{array}{ll}\text{for } y=0: & a)\ T=\tau(x) \quad \text{or } b)\ \partial_y T=\tilde{\tau}(x) \quad \text{or} \\ & c)\ \partial_y T+\Theta T=\tilde{\tau}(x).\end{array}\right\} \qquad (3)$$

Here $\alpha=0,1$, $x_0=0$; u, v are the components of the displacement vector along the x- and y-axes, respectively; T is a temperature change; σ_{xx}, σ_{yy} are normal stresses, $\sigma_{xy}=\sigma_{yx}$ are tangent stresses; Θ is a given constant; $\tau(x)$, $\tilde{\tau}(x)$ are given analytic functions on the segment $[0;\ x_1]$; $\partial_x \equiv \frac{\partial}{\partial x}$; $\partial_y \equiv \frac{\partial}{\partial y}$.

Note that the boundary conditions (1 a) are the antisymmetry conditions, and the boundary conditions (1 b) are the symmetry ones [6].

The problem is to give, on the side $y=y_1$, a temperature change T such that the following condition be fulfilled

$$v(x,y_2)-av(x,y_3)=g(x), \qquad (4)$$

where y_2 and y_3 are constants; without loss of generality it is assumed that $0<y_3<y_2<y_1$; a is some; $g(x)$ is an analytic function given on the segment $[0;x_1]$.

As is known, in the presence of mass forces we describe plane thermoelastic equilibrium by the following system of differential equations [8]

$$\begin{cases} \mu\Delta u + (\lambda+\mu)\,\partial_x\left(\partial_x u + \partial_y v\right) - \beta\partial_x T = 0, \\ \mu\Delta v + (\lambda+\mu)\,\partial_y\left(\partial_x u + \partial_y v\right) - \beta\partial_y T = 0, \end{cases} \tag{5}$$

where λ and μ are the Lame's constants; β is the linear thermal expansion coefficient.

A change of temperature T satisfies the two-dimensional plane Laplace equation

$$\Delta T = 0 \qquad \left(\Delta = \partial_{xx} + \partial_{yy}\right). \tag{6}$$

Stresses and displacements are related by the well-known Duhamel-Neumann formulas (see e.g. [8]).

3. Solution of the stated problems. A general solution of system (5) is represented through two harmonic functions φ and ψ. Let us write this solution omitting the details of its derivation:

$$2\mu u = \frac{\lambda+3\mu}{\lambda+\mu}\varphi + y\partial_y\varphi + \partial_y\psi + \frac{\mu\beta}{\lambda+\mu}T*, \tag{7}$$

$$2\mu v = -y\partial_x\varphi - \partial_x\psi + \frac{\mu\beta}{\lambda+\mu}\tilde{T}, \tag{8}$$

where T^* and $\tilde{T}$ are the harmonic functions related to the function T by

$$\partial_x T^* = T, \qquad \partial_y \tilde{T} = T, \qquad \partial_y T^* = -\partial_x \tilde{T}. \tag{9}$$

Stresses are expressed through the functions φ and ψ as follows

$$\begin{aligned} &\sigma_{xx} = \tfrac{2\lambda+3\mu}{\lambda+\mu}\partial_x\varphi + y\partial_{xy}\varphi + \partial_{xy}\psi, \quad \sigma_{yy} = -\tfrac{\mu}{\lambda+\mu}\partial_x\varphi - y\partial_{xy}\varphi - \partial_{xy}\psi, \\ &\sigma_{xy} = \tfrac{\lambda+2\mu}{\lambda+\mu}\partial_y\varphi + y\partial_{yy}\varphi + \partial_{yy}\psi, \quad \sigma_{zz} = \tfrac{\lambda}{\lambda+\mu}\partial_x\varphi - \tfrac{\mu\beta}{\lambda+\mu}T, \end{aligned} \tag{10}$$

where σ_{zz} is the normal stress which supports the plane deformed state.

Since the methods of construction of solutions of all the posed problems is the same, we will describe in detail only the solution of problem (5), (6), (1a), (2), (3a), (4).

Taking into account the boundary conditions (1a) and (2), from the general solutions (7), (8) and (10) we obtain $\varphi = \psi = 0$. Thus, from these general conditions we have

$$\begin{gathered} u = \frac{\beta}{2\left(\lambda+\mu\right)}T^*, \quad v = \frac{\beta}{2\left(\lambda+\mu\right)}\tilde{T}, \\ \sigma_{xx} = \sigma_{yy} = \sigma_{xy} = \sigma_{yz} = \sigma_{xz} = 0, \quad \sigma_{zz} = -\frac{\beta\mu}{\lambda+\mu}T. \end{gathered} \tag{11}$$

The analytic functions $\tau(x)$ and $g(x)$ are represented as the following Fourier series

$$\tau(x) = \sum_{m=1}^{\infty} \tau_m \sin\frac{\pi m x}{x_1}, \qquad g(x) = \sum_{m=1}^{\infty} g_m \sin\frac{\pi m x}{x_1}, \tag{12}$$

where

$$\tau_m = \mathrm{O}\left(e^{\gamma_m(y_3-y_1)}\right), \qquad g_m = \mathrm{O}\left(e^{-\gamma_m y_1}\right), \qquad \gamma_m = \frac{\pi m}{x_1}. \tag{13}$$

Conditions (13) guarantee the convergence of the resulting series in the domain $\bar{\omega}$.

Using the method of separation of variables and taking relations (9) and boundary conditions (1a) into account, we represent the harmonic functions T^* and $\tilde{T}$ as

$$\begin{aligned} T^* &= \sum_{m=1}^{\infty} -\tfrac{1}{\gamma_m}\left(A_m e^{-\gamma_m y} + B_m e^{\gamma_m y}\right)\cos\left(\gamma_m x\right), \\ \tilde{T} &= \sum_{m=1}^{\infty} \tfrac{1}{\gamma_m}\left(-A_m e^{-\gamma_m y} + B_m e^{\gamma_m y}\right)\sin\left(\gamma_m x\right). \end{aligned} \tag{14}$$

Thus the change of the temperature T is defined by the formula

$$T = \sum_{m=1}^{\infty}\left(A_m e^{-\gamma_m y} + B_m e^{\gamma_m y}\right)\sin\left(\gamma_m x\right). \tag{15}$$

Substituting the second expression (14) into the second formula (11), for the displacement v we have

$$v = \frac{\beta}{2\left(\lambda+\mu\right)}\sum_{m=1}^{\infty}\frac{1}{\gamma_m}\left(-A_m e^{-\gamma_m y} + B_m e^{\gamma_m y}\right)\sin\left(\gamma_m x\right). \tag{16}$$

Substituting expansions (15) and (16) into conditions (3a) and (4), respectively, inserting the respective series (12) in these conditions and further equating the coefficients of the identical trigonometric functions, for the sought coefficients A_m and B_m we obtain a system of two linear algebraic equations with two unknowns. This system is presented and examined below.

4. Discussion of the obtained results. The system mentioned at the end of the preceding subsection has the form

$$\begin{cases} A_m + B_m = \tau_m, \\ -\left(e^{-\gamma_m y_2} - a e^{-\gamma_m y_3}\right)A_m + \left(e^{\gamma_m y_2} - e^{\gamma_m y_3}\right)B_m = \frac{2\gamma_m(\lambda+\mu)}{\beta} g_m. \end{cases} \tag{17}$$

The conditions by which the determinant of system (17) is different from zero impose the following conditions on the coefficient a

$$a \neq \frac{\cosh\left(\gamma_m y_2\right)}{\cosh\left(\gamma_m y_3\right)}, \qquad m \in N. \tag{18}$$

If conditions (18) are fulfilled, then all the sought coefficients A_m and B_m are uniquely defined from system (17).

For the change of the temperature T we obtain the expression

$$T = \sum_{m=1}^{\infty}\frac{1}{\cosh\left(\gamma_m y_2\right) - a\cosh\left(\gamma_m y_3\right)}\left\{\left[\cosh\left(\gamma_m\left(y-y_2\right)\right) - a\cosh\left(\gamma_m\left(y-y_3\right)\right)\right]\tau_m\right.$$

$$+\frac{2\gamma_m(\lambda+\mu)}{\beta}\sinh(\gamma_m y)\,g_m\Bigg\}\sin(\gamma_m x)\,. \tag{19}$$

If τ_m and g_m satisfy conditions (13), then the obtained series (19) converges absolutely and uniformly in the domain $\bar{\omega}$ and, moreover, the obtained function T is analytic in this domain.

Replacing y by y_1 in (19), we obtain the sought temperature value of on the boundary $y = y_1$. This is a unique solution of the posed problem. It is not difficult to prove that the obtained solution will depend continuously on the initial data provided that the Fourier coefficients of the functions$\tau * (x)$, $g * (x)$, which are some disturbances of the functions $\tau(x)$, $g(x)$, also satisfy conditions (13).

Now assume that condition (18) is not fulfilled for some m_k of the index m. Then:

1) if the condition

$$\tau_{mn}+\frac{2(\lambda+\mu)\gamma_{m_k}}{\beta\left(e^{-\gamma_{m_k}y_2}-e^{-\gamma_{m_k}y_3}\right)}g_{m_k}=0$$

is fulfilled, then the stated problem has an infinite number of analytic solutions in $\bar{\omega}$;

2) if the latter equality is not fulfilled, then the stated problem has no solution.

REFERENCES

1. Khachikyan A.S. On harmonic and biharmonic problems for equations with nonclassical boundary conditions. (Russian) *Izv. Nats. Akad. Nauk Armen. Mekh.*, **59**, 4 (2006), 24-31.

2. Sarkisyan V.G. A plane elasticity problem with non-classical boundary conditions. Actual problems of continuum mechanics. *Proc. Intern. Conf. devoted to the 95-th anniversary of Academician NAN of Armenia N. Kh. Arutyunyan, Erevan*, (2007), 373-376.

3. Gordeziani D.G. A class of nonlocal boundary value problems in elasticity theory and shell theory. (Russian) *Theory and numerical methods of calculating plates and shells,* II, (1984), 106-127.

4. Obolashvili E.I. Solution of nonlocal problems in plane elasticity theory. (Russian) *Current problems in mathematical physics*, II, (1987), 295-302.

5. Bitsadze A.V. On the theory of nonlocal boundary value problems. (Russian) *Dokl. Akad. Nauk SSSR*, 2**77**, 1 (1984), 17-19.

6. Khomasuridze N.G. On the symmetry principle in continuum mechanics. (Russian) *Prikl. Mat. Mekh.*, **71**, 1 (2007), 23-32; translation in *J. Appl. Math. Mech.*, **71**, 1 (2007), 20-29.

7. Khomasuridze N. Thermoelastic equilibrium of bodies in generalized cylindrical coordinates, *Georgian Math. J.*, **5**, 6 (1998), 521-544 .

8. Muskhelishvili N.I. Some Basic Problems of the Mathematical Theory of Elasticity. *Noordhoff, Gröningen, Holland*, (1958).

Received 29.07.2011; revised 12.10.2011; accepted 25.11.2011.

Authors' address:

N. Khomasuridze, R. Janjgava, N. Zirakashvili
Iv. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia
E-mail: khomasuridze.nuri@gmail.com
roman.janjgava@gmail.com
natzira@yahoo.com