Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 25, 2011

ON SOME NEW ANALYTIC EXPRESSIONS OF COULOMB COLISSION THEORY FUNCTIONS

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Abstract. As it is known, the quantum mechanical formalism of two charge particles problem of continuous spectra is incomplete and inconsistent up to now. In present article we attempt to remove partially this distortion in Schrodinger formalism frame. There are proposed the main results of our research connected to the Coulomb quantum mechanical problem.

Keywords and phrases: Coulomb T-matrix, continious spectrum wave function fourier transporm.

AMS subject classification: 33C05, 42A16, 81U05.

The radial part of half-shell Coulomb T-matrix having outgoing asymptotics can be expressed as

$$\langle q|\ T_l^+(E)\ |k\rangle = \int\limits_0^\infty dr\, r\, j_l(qr) R_l^+(k\,r), \tag{1}$$

where k is kinematic parameter, q is absolute value of a vector in momentum space, $j_l(qr)$ denotes the spherical Bessel function, and $R_l^+(kr)$ is the outgoing Coulomb wave function (WF). In present article we have found explicit analytical expression of the matrix element (1). To obtain this expression one has to prove next

Theorem 1. *For any complex q and k, except the points $q = k$, one has:*

$$\int\limits_0^\infty dr\, r\, j_l(qr) R_l^+(k\,r) = \frac{(-1)^{i\gamma} 2^{2l} \exp(\pi\gamma/2)}{k^2}$$

$$\times \frac{\Gamma(l+1)}{\Gamma(l+i\gamma+1)}\, a^l \int\limits_0^1 dt\, \frac{t^{l+i\gamma}(1-t)^{l-i\gamma}}{\left[1-a^2(1-2\,t)^2\right]^{l+1}}, \tag{2}$$

$$(k, q \in C,\ a = q/k \neq 1),$$

where $\Gamma(x)$ is the Euler gamma function and γ denotes the Coulomb parameter, $\gamma = \pm 1/k$.

We have proved the Theorem 1 and use it to prove the

Theorem 2. *The radial part of half-shell Coulomb T-matrix depends on the parameters q, k as*

$$\langle q|T_l^+(E)|k\rangle = \frac{(-1)^{i\gamma}\exp(\pi\gamma/2)}{2kq}\,\frac{\Gamma(l-i\gamma+1)}{\Gamma(l+i\gamma+1)} Q_l^{i\gamma}\left(\frac{q^2+k^2}{2qk}\right), \qquad q \neq k. \tag{3}$$

Here $Q_l^{i\gamma}$ denotes the Legendre function of the second kind.

Using the Gauss hypergeometric function (HGF):

$$F(\alpha,\beta,\gamma;z) = F\left(\begin{array}{c}\alpha\\ \gamma\end{array};\beta;z\right)$$

the expression (2) can be rewritten as:

$$\begin{aligned}
&\langle q|\ T_l^+(E)\ |k\rangle \\
&\quad = \frac{\sqrt{\pi}\,(-1)^{i\gamma}\exp(\pi\gamma/2)}{2}\,\frac{\Gamma(l-i\gamma+1)}{\Gamma(l+3/2)}\,\frac{(qk)^l}{(k^2+q^2)^{l+1}}\left|\frac{k^2-q^2}{k^2+q^2}\right|^{i\gamma}\\
&\quad\times F\left(\begin{array}{c}1+(l+i\gamma)/2\\ l+3/2\end{array};(1+l+i\gamma)/2;\left(\frac{2qk}{k^2+q^2}\right)^2\right),\qquad q\neq k,\\
&\quad = (-1)^{i\gamma}(\sqrt{\pi}/2)\exp(\pi\gamma/2)\Gamma(l-i\gamma+1)\,\frac{(qk)^l}{(k^2+q^2)^{l+1}}\left|\frac{k^2-q^2}{k^2+q^2}\right|^{i\gamma}\\
&\quad\times F_R\left(\begin{array}{c}1+(l+i\gamma)/2\\ l+3/2\end{array};(1+l+i\gamma)/2;\left(\frac{2qk}{k^2+q^2}\right)^2\right),\qquad q\neq k.
\end{aligned}$$

Here F_R denotes the regularized HGF ([1], Eq. 15.1.(2)):

$$F_R(a,b;\,c;\,z) = F_R\left(\begin{array}{c}a\\ c\end{array};b;\,z\right) = \frac{1}{\Gamma(c)}F\left(\begin{array}{c}a\\ c\end{array};b;\,z\right).$$

Using the definition of the Fourier transform of radial part of the Coulomb WF

$$F_l(q,\,k) = \int_0^\infty dr\,r^2\,j_l(qr)R_l(k\,r) \tag{4}$$

we obtain explicit analytical expression for it. Here $R_l(kr)$ is radial part of the Coulomb WF in the coordinate representation (see [1], Eq. 14.1.(3)). Note that the radial part of the Coulomb WF differs from the function $R_l^+(kr)$ which has outgoing asymptotics.

To calculate the integral (4) previously we have proved next theorems:

Theorem 3. *For any complex q and k, except the points $q = k$ one has:*

$$\begin{aligned}
&\int_0^\infty dr\,r^2\,j_l(qr)R_l(k\,r) = i\frac{(-1)^{i\gamma}2^{2l+1}\exp(\pi\gamma/2)}{k^3}\\
&\qquad\times\frac{\Gamma(l+2)}{|\,\Gamma(l+i\gamma+1)\,|}\,a^l\int_0^1 dt\,\frac{t^{l+i\gamma}(1-t)^{l+i\gamma}(1-2t)}{\left[1-a^2(1-2\,t)^2\right]^{l+2}},\\
&\qquad(k,q\in C\quad a=q/k\neq 1)\,.
\end{aligned}$$

Theorem 4. *The radial part of Fourier transform depends on the parameters q, k as*

$$F_l(q,\,k) = \exp(3\pi\gamma/2)\frac{\Gamma(l-i\gamma+1)}{|\,\Gamma(l+i\gamma+1)\,|}\,\frac{\gamma(-1)^{i\gamma}}{|q\,(k^2-q^2)|}Q_l^{i\gamma}\left(\frac{k^2+q^2}{2qk}\right),\qquad q\neq k,$$

or, in alternative forms

$$
\begin{aligned}
&F_l(q,\,k)\\
&= \sqrt{\pi}\,(-1)^{i\gamma}\exp(\pi\gamma/2)\;\frac{|\,\Gamma(l+i\gamma+1)\,|}{\Gamma(l+3/2)}\,\frac{\gamma k}{(k^4-q^4)}\left(\frac{qk}{k^2+q^2}\right)^l\left|\frac{k^2-q^2}{k^2+q^2}\right|^{i\gamma}\\
&\times F\left(\begin{matrix}1+(l+i\gamma)/2\\ l+3/2\end{matrix};(1+l+i\gamma)/2;\left(\frac{2qk}{k^2+q^2}\right)^2\right),\qquad q\neq k.\\
&= \sqrt{\pi}\,(-1)^{i\gamma}\exp(\pi\gamma/2)\;|\,\Gamma(l+i\gamma+1)\,|\,\frac{\gamma k}{(k^4-q^4)}\left(\frac{qk}{k^2+q^2}\right)^l\left|\frac{k^2-q^2}{k^2+q^2}\right|^{i\gamma}\\
&\times F_R\left(\begin{matrix}1+(l+i\gamma)/2\\ l+3/2\end{matrix};(1+l+i\gamma)/2;\left(\frac{2qk}{k^2+q^2}\right)^2\right),\qquad q\neq k.
\end{aligned}
$$

Representations (1) and (4) express quantum-mechanical functions in terms of the Legendre functions of second kind $Q_\nu^\mu(z)$ having cut in the complex plane along the segment $z \in [-1,+1]$. This singularity specifies indefiniteness of the functions at the point $z = 1$ corresponding to elastic scattering [2]. By-turn, S-matrix formalism of perturbations theory combined with our investigations gives possibility to present the matrix element (3) analytically at the point $z = 1$ and to obtain for this function correct expression everywhere in the right half plane $Req > 0, Rek > 0$ of complex momentum space, including physical region $Imq = Imk = 0$. This result gives an improvement of the Ford rule [3] and a generalization of the rule for any (complex) values of angular momentum l (except, may be, some poles on negative part of real axes). To our opinion, constructed two charge particle regular T-matrix, as well as constructed Coulomb WF Fourier transform makes the theory more consistent and complete.

R E F E R E N C E S

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Received 23.06.2011; revised 21.09.2011; accepted 23.11.2011.

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