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ON THE FOURIER COEFFICIENTS OF CONTINUOUS FUNCTIONS WITH RESPECT TO GENERAL ORTHOGONAL SYSTEMS (ONS)

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Abstract. As it is well known (see.[1], p. 119, [2], [3]) for the classical orthonormal systems (trigonometric system, Haar system and Walsh system), the Fourier coefficients of continuous functions are estimated from above by modulus of continuity of these functions. For general ONS , however (see.[4]),we have, if $(a_n) \in l_2$ is any sequence and $f(x) \in C(0,1)$ any function, there exists ONS such that the Fourier coefficients of the function $f(x)$ are estimated from above, respectively, by the numbers $a_n (n = 1, 2, ...)$. It follows that in the general case the Fourier coefficients of the function $f(x) \in C(0,1)$ are not estimated by the modulus of continuity of these functions.

The main purpose of the present paper is to find the conditions which should be satisfied by the functions of ONS $(\varphi_n(x))$ and under which the Fourier coefficients of functions $f(x) \in C(0,1)$ and $f' \in L_2(0,1)$ be estimated from above by the modulus of continuity or smoothnes of these functions.

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Notation and auxiliary theorems:

Let $f(x) \in C(0,1)$, then

$$\omega\Big(\frac{1}{n}, f\Big) = \sup_{|x-y| \leq \frac{1}{n}} | f(x) - f(y) |,$$

is the modulus of continuity of the function $f(x)$.

Besides, if $f(x) \in C(0,1)$, then

$$\omega^{(2)}\Big(\frac{1}{n}, f\Big) = \sup_{|x-y| \leq \frac{1}{n}} | f(x) - 2f\Big(\frac{x+y}{2}\Big) + f(y) |,$$

is the modulus of smoothnes of the function $f(x)$.

The value

$$\omega_2\Big(\frac{1}{n}, f\Big) = \sup_{|h| \leq \frac{1}{n}} \Big(\int_0^{1-h} |f(x+h) - f(x)|^2\Big)^{\frac{1}{2}},$$

is said to be an integral modulus of continuity.

Let $(\varphi_n(x))$ be a give ONS

$$Q_{np} = \sum_{k=n}^{n+p} a_k \varphi_k(x),$$

$$B_{np} = \sum_{i=1}^{n-1} \mid \int_0^{\frac{i}{n}} Q_{np} dx \mid,$$

$$S_{np}(x) = \int_0^x Q_{np}(y)dy,$$

$$H_{np} = \sum_{i=1}^{n-1} \mid \int_0^{\frac{i}{n}} S_{np} dx \mid,$$

$$\widehat{\varphi}_n(f) = \int_0^1 f(x) f_n(x) dx, \quad n = 1, 2, ...,$$

are Fourier coefficients of the function $f(x)$.

If $f(x) \in L_2(0,1)$, then $e_n(f) = \Big(\sum_{k=n}^{\infty} \widehat{\varphi}^2(f) \Big)^{\frac{1}{2}}$.

Let $(a_k) \in l_2$ be some sequence of numbers, $e_n = \Big(\sum_{k=n}^{\infty} a_k^2 \Big)^{\frac{1}{2}}$.

Main results:

Theorem 1. *Let $f_n(x)$ be ONS on $[0,1]$ and $f^{'}(x) \in L_2(0,1)$. Then for any natural p the inequality holds.*

$$e_n(f) = O\Big(\omega^{(2)}\Big(\frac{1}{n}, f\Big) H_{np} + \Big(\omega_2\Big(\frac{1}{n}, f^{'}\Big) + \omega\Big(\frac{1}{n}, f\Big)\Big) B_{np} \Big).$$

Theorem 2. *Let $\varphi_n(x)$ ONS on $[0,1]$ such that $\int_0^1 \varphi_n(x)dx = 0$ and $\int_0^1 \int_0^x \varphi_n(t)dtdx = 0$ and for any sequence of numbers*

a) $H_{np} = O(e_n)$,

b) $B_{NP} = O(e_n)$ are satisfied.

Then for any function $f^{'}(x) \in L_2(0,1)$, $e_n(f) = O\Big(\omega^{(2)}\Big(\frac{1}{n}, f\Big) H_{np} + \omega_2\Big(\frac{1}{n}, f^{'}\Big) \Big)$.

Theorem 3. *Let $f_n(x)$ be ONS on $[0,1]$, $\int_0^1 \varphi_n(x)dx = 0$ and $\int_0^1 \int_0^x \varphi_n(t)dtdx = 0$. Then if $H_{np} \neq O(e_n)$ or $B_{np} \neq O(e_n)$, for some sequence $(a_n) \in l_2$, there exists the function $g(x)$ such $g^{'}(x) \in L_2(0,1)$ and*

$$e_n(g) \neq O\Big(\omega^2\Big(\frac{1}{n}, g\Big) \Big).$$

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