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RELATION BETWEEN BELTRAMI AND HOLOMORPHIC DISC EQUATIONS

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Abstract. In this paper we give detailed analysis pseudo-analytic functions theory point of view Beltrami and holomorphic disc equations and prove the equivalence this equations.

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The objects of study of this paper are particular cases of the general elliptic system: Beltrami [1], [2] and holomorphic disc equations [3]. The paper is continuation of the work of first author [4], detailed definition we don't give here and sometime directly give refer to [4].

Let (F, G) normalized generating pair on complex space $\mathbb{C}$ [5] it means that 1) $F, G \in C_{\frac{p-2}{p}}\mathbb{C}$, $p > 2$; 2)$F_{\overline{z}}, G_{\overline{z}} \in L_{p,2}(\mathbb{C}) \bigcap C_\beta(\mathbb{C}^{loc})$, $0 < \beta < 1$; 3)$Im(\overline{F}(z)G(x)) \geq K_0 > 0$, $K_0 = const$, $z \in \mathbb{C}$. Every function W, at every points, unique represented by $F(z), G(z)$ in following form

$$W(z) = \varphi(z)F(z) + \psi(z)G(z),$$

where φ, ψ real functions.

Let $W(z)$ is (F, G)-pseodoanalytic in $\mathbb{C}$, then it is know that $W(z)$ is solution of Carlemann-Vekua equation

$$W_{\overline{z}} = AW + B\overline{W},$$

where

$$A = \frac{F_{\overline{z}}\overline{G} - \overline{F}G_{\overline{z}}}{F\overline{G} - \overline{F}G}, B = \frac{G_{\overline{z}}F - F_{\overline{z}}G}{F\overline{G} - \overline{F}G}.$$

From the pseodo-analytic follows also, that there exist continuations partial derivalives $\varphi_z, \varphi_{\overline{z}}, \psi_z, \psi_{\overline{z}}$ and

$$F\varphi_{\overline{z}} + G\psi_{\overline{z}} = 0.$$

Consider the function

$$\omega(z) = \varphi(z) + i\psi(z).$$

Then

$$2(F\varphi_{\overline{z}} + G\psi_{\overline{z}}) = (F - iG)(\varphi_{\overline{z}} + i\psi_{\overline{z}}) + (F + iG)(\varphi_{\overline{z}} - i\psi_{\overline{z}}) =$$

$$= (F - iG)(\varphi + i\psi)_{\overline{z}} + (F + iG)(\varphi - i\psi)_{\overline{z}} = (F - iG)\omega_{\overline{z}} + (F + iG)\overline{\omega}_{\overline{z}} = 0.$$

From this follows, that

$$\omega_{\overline{z}}(F - iG) + \overline{\omega_z}(F + iG) = 0. \tag{1}$$

Lemma 1. $F(z) - iG(z) \neq 0$ Indeed,

$$|F(z) - iG(z)|^2 = (F(z) - iG(z)(\overline{F(z) - iG(z)}) = (F(z) - iG(z)(\overline{F(z)} + i\overline{G(z)}) =$$

$$= |F(z)|^2 + |G(z)|^2 - i(\overline{F(z)}G(z) - F(z)\overline{G(z)}) =$$

$$= |F(z)|^2 + |G(z)|^2 + 2Im(\overline{F(z)}G(z)) \geq 2K_0 > 0,$$

when $|F(z)|^2 > 0, |G(z)|^2 > 0, Im(\overline{F(z)}G(z)) \geq K_0$ for every $z \in \mathbb{C}$. Lemma proved.

From Lemma 1 and (1) follows, that

$$\Rightarrow \omega_{\overline{z}} + \overline{\omega_z}\frac{F + iG}{F - iG} = 0.$$

Denote by $q(z) = -\frac{F(z)+iG(z)}{F(z)-iG(z)}$.

Lemma 2. $|q(z)| \leq q_0 < 1$, $z \in \mathbb{C}$

Step 1.

$$|q(z)|^2 = \frac{|F(z) + iG(z)|^2}{|F(z) - iG(z)|^2} = \frac{(F(z) + iG(z))\overline{(F(z) + iG(z))}}{(F(z) - iG(z))\overline{(F(z) - iG(z))}} \Rightarrow$$

$$\Rightarrow \frac{|F(z)|^2 + |G(z)|^2 - 2Im(\overline{F(z)}G(z))}{|F(z)|^2 + |G(z)|^2 + 2Im(\overline{F(z)}G(z))} < 1, \tag{2}$$

when $Im(\overline{F(z)}G(z)) \geq K_0 > 0$, $z \in \mathbb{C}$.

Step 2. The function F, G satisfies Carlemnn-Vekua equation

$$F_{\overline{z}} = aF + b\overline{F}, G_{\overline{z}} = aG + b\overline{G}, \tag{3}$$

when $F \in C_{\frac{p-1}{p}}(\mathbb{C})$, $a, b \in L_{p,2}(\mathbb{C})$ we obtain $aF + b\overline{F} \in L_{p,2}(\mathbb{C})$. From (3) follows, that

$$F(z) = \Phi(z) + T_{\mathbb{C}}(aF + b\overline{F})(z), \tag{4}$$

where $\Phi(z)$ entire function. From $F(z), T_{\mathbb{C}}(aF+b\overline{F})(z) \in C_{\frac{p-2}{p}}(\mathbb{C})$, follows that $\Phi(z) \in C_{\frac{p-2}{p}}(\mathbb{C})$. By Liuvile theorem we obtain $\Phi(z) = const$, therefore $\Phi(z) = C, z \in \mathbb{C}$. From this and (4) obtain

$$F(z) = C + T_{\mathbb{C}}(aF + b\overline{F})(z). \tag{5}$$

When $T_{\mathbb{C}}(aF+b\overline{F})(\infty) = 0$, from (5) follows, that $F(\infty) = C$. In similar way we obtain $G(\infty) = C_1$.

When $Im(\overline{F(z)}G(z)) \geq K_0$, therefore

$$Im(\overline{F(\infty)}G(\infty)) \geq K_0. \tag{6}$$

From (2) and (6) $\Rightarrow$

$$\Rightarrow |q(\infty)|^2 = \frac{|F(\infty)|^2 + |G(\infty)|^2 - 2Im(\overline{F(\infty)}G(\infty))}{|F(\infty)|^2 + |G(\infty)|^2 + 2Im(\overline{F(\infty)}G(\infty))} < 1. \tag{7}$$

From (2) and (7) follows, that

$$|q(z)| < 1, z \in \mathbb{C}, |\text{ıı}(\infty)| < \nVdash,$$

therefore $|q(z) \leq q_0 < 1,\ z \in \mathbb{C}$.

Proposition 1. *The exist the function* $\widetilde{q}(z)$, *such that* ω *is the solution of Beltrami equation with coefficient* $\widetilde{q}(z)$.

Introduced the function $\widetilde{q}(z)$:

$$\widetilde{q}(z) = \begin{cases} q(z)\dfrac{\overline{\partial_z \omega}}{\partial_z \omega}, \text{when } \partial_z \omega \neq 0, \\ 0, \text{when } \partial_z \omega = 0. \end{cases} \tag{8}$$

and consider the equation

$$\partial_{\overline{z}} \omega - q(z)\frac{\overline{\partial_z \omega}}{\partial_z \omega} = 0.$$

From (8) follows that ω satisfies the equation

$$\partial_{\overline{z}} \omega - \widetilde{q(z)} \partial_z \omega = 0. \tag{9}$$

It is clear, that

$$|\widetilde{q(z)}| = |q(z)\frac{\overline{\partial_z \omega}}{\partial_z \omega}| = |q(z)||\frac{\overline{\partial_z \omega}}{\partial_z \omega}| = |q(z)| \leq q_0 < 1. \tag{10}$$

From (9) and (10) follows, that $\omega(z)$ is solution of the Beltrami equation

$$\partial_{\overline{z}} h - \widetilde{q(z)} \partial_z h = 0. \tag{11}$$

In area $U \subset \mathbb{C}$ the function ω represented as $\omega(z) = \Psi(W(z))$, where $W(z)$ is complete homeomorphism of the equation (11) and $\Psi(\zeta)$ analytic on $W(U)$ function.

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