Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 25, 2011

ON THE STRUCTURE OF THE SPACE OF GENERALIZED ANALYTIC FUNCTIONS

Giorgadze G.

Abstract. In this paper we investigate relation between the holomorphic and conformal structures and induced from conformal structures spaces of generalized analytic functions.

Keywords and phrases: Conformal structure, generalized analytic function.

AMS subject classification: 30G20.

Introduction. The general first order elliptic system of partial differential equation

$$\frac{\partial w(z)}{\partial \bar{z}} - \mu_1(z)\frac{\partial w(z)}{\partial z} - \mu_2(z)\frac{\overline{\partial w}}{\partial z} = A(z)w(z) + B(z)\overline{w(z)} + C(z), \tag{1}$$

on the complex plane with natural restriction of the functions μ_1, μ_2, A, B contains many well know equations: Coushy-Riemann, Beltrami, Carleman-Bers-Vekua, holomorphic disc and other equations, which obtained from (1) after appropriate chouse the coefficients. All this equations are "deformation" of Coushy-Riemann equation and the properties of the solutions spaces of corresponding equations are near the properties of the spaces of analytic functions.

Below we consider relation between induced from complex structure solutions spaces of the following equations:

a) Carlemann-Bers-Vekua equation [1]

$$\frac{\partial w(z)}{\partial \bar{z}} = A(z)w(z) + B(z)\overline{w(z)};$$

b) Belrtami equation [2]

$$\frac{\partial w(z)}{\partial \bar{z}} = \mu_1(z)\frac{\partial w(z)}{\partial z};$$

c) Holomorphic disc equation [3]

$$\frac{\partial w(z)}{\partial \bar{z}} = \mu_1(z)\frac{\overline{\partial w(z)}}{\partial z}.$$

This equations are invariant respect to conformal transformations and therefore are corrected defined on Riemann surfaces. The function A, B defines the pair of complex functions (F, G), satisfied the inequality $Im(\overline{F}G) > 0$ and (F, G)-speodo-analytic are the solutions Carlemann-Bers-Vekua equation and visa versa [4].

Allmost complex structure. Let X two dimensional connected smooth manifold. By definition two complex atlases U and V are equivalent if their union is a complex atlas. A complex structure on X is an equivalence classes of complex atlases. A Riemann surface is a connected surface with a complex structure. Differential 1-form on X respect to local coordinate z presented in form $\omega = \alpha dz + \beta d\bar{z}$. Therefore, ω have bidegree (1,1) and is sum of the forms $\omega^{1,0} = \alpha dz$ and $\omega^{0,1} = \beta d\bar{z}$ of bidegree (1,0) and (0,1)respectively. The change of local coordinate $z \to iz$ induces on the map of differential forms and $\omega \to i(\alpha dz - \beta d\bar{z}) = i\omega^{1,0} - i\omega^{0,1}$. Denote by J the operator defined on 1-forms by the rule $J\omega = i\omega^{1,0} - i\omega^{0,1}$. This operator don't depends on the change of local coordinate z and $J^2 = -1$, where 1 denotes identity operator. Therefore, the splitting $\Lambda^1 = \Lambda^{1,0} + \Lambda^{0,1}$ is decomposition of space of differential 1-forms by proper subspaces of $J : T^*(X)_{\mathbb{C}} \to T^*(X)_{\mathbb{C}}$. On the tangent space TX the operator J acts as $\omega(Jv) = (J\omega)(v)$, for every vector field $v \in TX$. If $z = x + iy$ and take $v = \frac{\partial}{\partial x}$, then

$$dz(Jv) = idz\left(\frac{\partial}{\partial x}\right) = i = dz\left(\frac{\partial}{\partial y}\right) \Rightarrow J\frac{\partial}{\partial x} = \frac{\partial}{\partial y},\ J\frac{\partial}{\partial y} = -\frac{\partial}{\partial x}.$$

It means, that on the basis $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ of TX the operator J is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Therefore, by the complex structure defined from local coordinates defines the operator $J : T^*(X)_{\mathbb{C}} \to T^*(X)_{\mathbb{C}}$, whit properties $J^2 = -1$. This operator called pseodo complex structure.

On contrary, let X smooth surface and let $J : T_x(X) \to T_x(X)$, $x \in X$ such operator, that $J^2 = -1$. The pare (X, J) called pseodo analytic surface. As above, by duality it is possible define J on 1-forms on X. The the space of one forms Λ^1 decomposed by proper subspaces correspondigs of eigenvalues $\pm i$ of J and $\Lambda^1 = \Lambda_J^{1,0} + \Lambda_J^{0,1}$. In particular, $J\Lambda_J^{1,0} = i\Lambda_J^{1,0}$ and $J\Lambda_J^{0,1} = -i\Lambda_J^{0,1}$.

Let f smooth function, then $df \in \Lambda^1$ and decomposed by bidegree $df = \partial_J f + \bar{\partial}_J f$, where $\partial_J f := (df)_J^{1,0}$ and $\bar{\partial}_J f := (df)_J^{1,0}$. By definition, f is J-holomorphic, if its satisfies the Coushy-Riemann equation $\bar{\partial}_J f$.

Let (X, J) pseodo complex surface. In the neighborhood of every point $x \in X$ possible change the local coordinate such, that dz will be $(1,0)_J$-type. Then decomposition of dz by bidegree is $dz = \omega + \bar{\delta}$, where ω, δ are forms of bidegree $(1,0)_J$. Because, the fibre of $T_J^{1,0}X$ is one dimension complex space, we have, that $\delta = \mu\bar{\omega}$, where μ some smooth function $\mu(0) = 0$. From this follows, that

$$dz = \omega + \mu\bar{\omega} \text{ and } d\bar{z} = \bar{\omega} + \bar{\mu}\omega. \tag{2}$$

Therefore, for every smooth function f in the neighborhood of $x \in X$ we have

$$df = (\partial f + \bar{\mu}\bar{\partial} f)\omega + (\bar{\partial} f + \mu\partial f)\bar{\omega} = \partial_J f + \bar{\partial}_J f$$

From this follows, that f is J-holomorphic iff $\bar{\partial}_J f = 0$, i.e.

$$\bar{\partial} f + \mu\partial f = 0.$$

The equation (1) called Beltrami equation. Therefore, the smooth functions, defined on (X, J) pseodo complex surface are J-holomorphic, iff satisfies Beltrami equation (1).

Suppose f is J-holomorphic and let $f = \varphi + i\psi$, where φ and ψ real valued functions. Consider the complex valued function w defined by identity $w = \varphi F + \psi G$, where F, G complex valued Holder continuous functions satisfies the condition $Im(\overline{F}G) > 0$.

Theorem. *The function $w = \varphi F + \psi G$ is (F, G)-pseudo-analytic.*
Indeed,$w = \varphi F + \psi G = \frac{iG-F}{2} f + \frac{-iG-F}{2}\overline{f}$, from this follows, that f is solution of the beltrami equation

$$(iG - F)\overline{\partial f} - (iG + F)\partial f = 0$$

iff w is solution of Carlemann-Bers-Vekua equation

$$\overline{\partial} w + \frac{\overline{F}\overline{\partial} G - \overline{\partial} F \overline{G}}{F\overline{G} - \overline{F}G} w + \frac{F\overline{\partial} G - \overline{\partial} F G}{F\overline{G} - \overline{F}G} \overline{w} = 0.$$

In $D \subset \mathbb{C}$ every metric have the form $\lambda|dz+\mu d\overline{z}|$, where $\lambda > 0$ and complex function μ satisfies properties $|\mu| < 1$, from this follows, that J it is defined in unique ways by 1-form $\omega = dz+\mu d\overline{z}$ on D with properties $J\omega = i\omega$, $J\overline{\omega} = -i\omega$. The forms of this type are (1,0) bidegree forms respect to J (we have denoted by $\Lambda_J^{1,0}$. If $\delta \in \Lambda_J^{1,0}$, then $\delta = \alpha\omega+\beta\overline{\omega}$ and it is proportional to ω. Therefore J defines unique with accuracy constant multiplier $(1, 0)_J$ form ω. Holomorphic respect to J functions have proportional to ω differentials. Indeed, if $df + iJ(df) = 0$, then $J(df) = idf$ and from the representation $df = \alpha\omega + \beta\overline{\omega}$ obtain, that $\beta\overline{\omega} = 0$. Because $df = \alpha\omega + \beta\overline{\omega}$, therefore in $D \subset \mathbb{C}$ Coushy-Riemann equation respect to J with base form $\omega = dz + \mu d\overline{z}$ represented as Beltrami equation $\overline{\partial} f = \mu\partial f$. This equation have such solution f, that it is biholomorphic map from (D, J) to $f(D), J_{st}$, where J_{st} standard conformal structure on $\mathbb{C}$.

Therefore we proved the following proposition.

Proposition 1. *In simple connected areas exists only one complex structure and conformal structures are one-to-one correspondence to complex functions μ, with $|\mu| <$* 1. From this proposition and theorem 1 follows proposition.

Proposition 2. *There exist one-to-one correspondence between spaces of conformal structures and space of generalized analytic functions on the connected open area of complex plane.*

The equation of holomorphic discs. Let $\mathbb{D}$ unity disc in complex plane $\mathbb{C}$ with standard complex structure J_{st} and coordinate function ζ. J_{st} unique it is define by form $d\zeta \in \Lambda_{J_{st}}^{1,0}$. The map $\phi : \mathbb{D} \to \mathbb{X}$ of the class C^1 is holomorphic iff $\psi^*\Lambda_J^{1,0}(X) \subset \Lambda^{1,0}(\mathbb{D})$. Let coordinate function on $\mathbb{D}$ is z. We study local problem, therefore, without restricted of generality, it is possible to consider ϕ as mapping from $(\mathbb{D}, \mathbb{J}_{st})$ to $(\mathbb{C}_z, \mathbb{J})$, where the complex structure J it is defined by $dz = \omega + \mu\overline{\omega}$, $\overline{\omega} \in \Lambda_J^{1,0}$. Therefore we have

$$\zeta \to z = z(\zeta), z(0) = 0.$$

From (2) we obtain that

$$\omega = \frac{dz - \mu d\overline{z}}{1 - |\mu|^2}.$$

The form ω is J-holomorphic means, that the form

$$z^*(dz - \mu d\overline{z}) = (\partial_\zeta z - \mu\partial_\zeta \overline{z})d\zeta + (\partial_{\overline{\zeta}} z - \mu\partial_{\overline{\zeta}}\overline{z})d\overline{\zeta}$$

have bidegree $(1,0)$ on $\mathbb{D}$, therefore

$$\partial_{\bar{\zeta}} z - \mu \partial_{\bar{\zeta}} \bar{z} = 0.$$

From this after used the identity $\partial_{\bar{\zeta}} \bar{z}$ obtain, that

$$\partial_{\bar{\zeta}} z = \mu(z) \overline{\partial_{\zeta}} z.$$

The obtained expression called *equation of holomorphic disc.* It is known, that f satisfies this equation iff f^{-1} satisfies corresponding Beltrami equation (see [5]).

Proposition 3. *If* $\omega = u + iv$ *satisfies the equation* $\frac{\partial w(z)}{\partial \bar{z}} + \mu(z) \overline{\frac{\partial w(z)}{\partial z}} = 0$, $|\mu| < 1$ *and* a *and* b *such holomorphic functions that* $\mu = \frac{a-b}{a+b}$, *then* $W = au + ibv$ *is holomorphic.*

Indeed,

$$\frac{\partial}{\partial \bar{z}} \left(a \frac{\omega + \overline{\omega}}{2} + ib \frac{\omega - \overline{\omega}}{2} \right) = \frac{a}{2} \left(\omega_{\bar{z}} + \overline{\omega}_{\bar{z}} \right) + \frac{b}{2} \left(\omega_{\bar{z}} - \overline{\omega}_{\bar{z}} \right) = \omega_{\bar{z}} \left(\frac{a+b}{2} \right) + \overline{\omega}_{\bar{z}} \left(\frac{a-b}{2} \right),$$

therefore if ω is solution of the equation $\omega_{\bar{z}} + \frac{a-b}{a+b} \overline{\omega_z} = 0$, then $\partial_{\bar{z}} W = 0$.

From this propositions in particular follows, that W is (a, ib)-pseudo-analytic.

Acknowledgements. The author is grateful to the participants of seminar "Elliptic systems on Riemann surfaces" in Tbilisi State University.

This paper was partially supported by the Rustaveli Foundation, Grant No 1-3/85.

REFERENCES

1. Vekua I. N. Generalized Analytic Functions. *2nd Edition, Nauka*, 1988.

2. Bojarski B. Generalized solutions of the PD system of the first order and elliptic type with discontinuous coefficients. *Mat. Sbornik.*, **43** (1957), 451-503.

3. Chirka E.M. Riemann Surfaces. *Steklov Math. Inst. Moscow*, 2006.

4. Bers L. Theory of Pseudo-analytic Functions. *New York University*, 1953.

5. Giorgadze G., Jikia V. Relation between Beltrami and holomorphic disc equations. *Rep. Enlarged Sess. Semin. I.Vekua Appl. Math.*, **25** (2011), 57-60.

Received 23.06.2011; revised 27.09.2011; accepted 30.11.2011.

Author's address:

Giorgadze G.
Department of Mathematics of
Iv. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia
E-mail: gia.giorgadze@tsu.ge