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ON THE DARBOUX TRANSFORMATION FOR CARLEMAN-BERS-VEKUA SYSTEM

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Abstract. In this paper we used Darboux transformation technique for investigation Stationary Schrödinder two dimensinal equation and s.c. main Vekua equation.

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The basic fact and definitions. The theory of pseudoanalytic functions have the goal of applying complex analysis methods to systems of partial differential equations which are more general that Caushy-Riemann systems [1], [2]. Recently in [3] give new application of the theory of pseudoanalytic functions to differential equations of mathematical physics.

The canonical form of a uniformly elliptic linear first-order system for two desired real-valued functions in a domain of the complex plane has the form

$$w_{\overline{z}} = a(z)w + b(z)\overline{w}, \tag{1}$$

which is know as Carleman-Bers-Vekua system. If f is a real valued function, then

$$w_{\overline{z}} = \frac{f_{\overline{z}}}{f}\overline{w} \tag{2}$$

is called the corresponding main Vekua equation. In [3] author's applications of pseudoanalytic functions to differential equations of mathematical physics are based on the factorization of a second order differential operator in a product of two first order differential operators whose one of these two factors leads to a main Vekua equation. In particular it is show, that if f, h, ψ are real-valued functions, $f, \psi \in C^2(\Omega)$, $\Omega \subset \mathbb{C}$ and besides f is positive particular solution of the two dimensional stationary Schrodinger equation

$$(-\Delta + h)f = 0 \tag{3}$$

in domain $\Omega \subset \mathbb{C}$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is two dimensional Laplase operator, then

$$(\Delta - h)\psi = 4(\partial_{\overline{z}} + \frac{f_z}{f}C)(\partial_{\overline{z}} - \frac{f_z}{f}C)\psi = 4(\partial_z + \frac{f_{\overline{z}}}{f}C)(\partial_z - \frac{f_{\overline{z}}}{f}C)\psi, \tag{4}$$

where C denotes the operator of complex conjugation.

Let $w = w_1 + iw_2$ be a solution of the equation (2). Then the functions $u = f^{-1}w_1$ and $v = fw_2$ are the solutions of the following conductivity and associated conductivity equations

$$div(f^2\nabla u) = 0, \text{ and } div(f^2\nabla u) = 0, \tag{5}$$

respectively. The real and imaginary part of the solution of the equation (2) w_1 and w_2 are solutions of the stationary Schrodinger and associated stationary Schrödinger equations

$$-\Delta w_1 + r_1 w_1 = 0 \quad \text{and} \quad -\Delta w_2 + r_2 w_2 = 0, \tag{6}$$

respectively, where $r_1 = \frac{\Delta f}{f}$, $r_2 = \frac{2(\nabla f)^2}{f^2} - r_1$, $\nabla f = (f_x, f_y)$ and $(\nabla f)^2 = f_x^2 + f_y^2$.

In other hand it is known that the elliptic equation

$$\partial_z \partial_{\overline{z}} \psi + h\psi = 0 \tag{7}$$

is covariant with respect to the Darboux transformation [4]

$$\psi \to \psi[1] = \theta(\psi, \psi_1)\psi_1^{-1}, \quad \theta(\psi, \psi_1) = \int_{(z_0,\overline{z}_0)}^{(z,\overline{z})} \Omega, \tag{8}$$

$$h[1] = h + 2\partial_z \partial_{\overline{z}} ln\psi_1, \tag{9}$$

where ψ_1 is a fixed solution of equation (7) and Ω is closed 1-differential form

$$\Omega = (\psi\partial_z\psi_1 - \psi_1\partial_z\psi)dz - (\psi\partial_{\overline{z}}\psi_1 - \psi_1\partial_{\overline{z}}\psi)d\overline{z}.$$

Here covariant properties means, that $\psi[1]$ satisfies the following equation

$$\partial_z \partial_{\overline{z}} \psi[1] + h[1]\psi[1] = 0.$$

From the equality $d\Omega = 0$ follows, that the function $\theta(\psi_1, \psi)$ in (8) does not depend on path of integration.

Main result.

Theorem 1. *Let $w = w_1 + iw_2$ is the solution of the main Vekua equation*

$$w_{\overline{z}} = \frac{\psi_{1\overline{z}}}{\psi_1}\overline{w}. \tag{10}$$

Then $w_1 = \psi_1$ and $w_2 = -\frac{1}{2}\psi[1]$, where ψ_1 is the real positive solution of the equation

$$-\Delta\psi + h\psi = 0, \tag{11}$$

$h = \frac{\triangle\psi_1}{\psi_1}$ and $\psi[1]$ its Darboux transformation defined by (8), (9).

Conversely, if ψ is the real positive solution of the equation (11) and $\psi[1]$ its Darboux transformation, then the solution of main Vekua equation (10) equal to $w = \psi_1 - i\frac{1}{2}\psi[1]$.

First part of the theorem follows from (6). Here we prove the second part of theorem. Let ψ is real solution of (11), then in this case the Darboux transformation (8),(9) has the form

$$h[1] = h - 2\Delta ln\psi_1 \quad \text{and} \quad \psi[1] = 2i\psi_1^{-1} Im \int (\psi\psi_{1\overline{z}} - \psi_{\overline{z}}\psi_1)d\overline{z}.$$

We seek the solution of the equation (10) in the form $w = \psi + iw_2$, where ψ is solution of (11). Then

$$\psi_{1\overline{z}} + iw_{2\overline{z}} = \frac{\psi_{1\overline{z}}}{\psi_1}\psi - i\frac{\psi_{1\overline{z}}}{\psi_1}w_2,$$

from this the solution of the corresponding homogenous equation is $w_2 = \frac{C(z)}{\psi_1}$, where $C(z)$ is arbitrary holomorphic function. Let $w_2 = \frac{C(z,\overline{z})}{\psi_1}$ be a solution of above equation. Then

$$\psi_{\overline{z}} + i\frac{C_{\overline{z}}}{\psi_1} - i\frac{\psi_{1\overline{z}}}{(\psi_1)^2}C(z,\overline{z}) = \frac{\psi_{1\overline{z}}}{\psi_1}\psi - i\frac{\psi_{1\overline{z}}}{(\psi_1)^2}C(z,\overline{z}) \Rightarrow$$

$$\Rightarrow C_{\overline{z}} = -i(\psi\psi_{1\overline{z}} - \psi_{\overline{z}}\psi_1) \Rightarrow C(z,\overline{z}) = -i\int(\psi\psi_{1\overline{z}} - \psi_{\overline{z}}\psi_1)d\overline{z} + \widetilde{C}(z).$$

From this we obtain that

$$w_2 = \psi_1^{-1}(b(z) - i\int(\psi\psi_{1\overline{z}} - \psi_{\overline{z}}\psi_1)d\overline{z}).$$

We chouse $b(z)$ in last expression such, that w_2 was real. Then $w_2 = \psi_1^{-1}Im\int(\psi\psi_{1\overline{z}} - \psi_{\overline{z}}\psi_1)d\overline{z}$, from this follows, that $-2iw_2 = \psi[1]$, therefore $w = \psi_1 - i\frac{1}{2}\psi[1]$ is the solution of (10).

Here we give new formulation and proof of Theorem 1.

Theorem 2. *1)Let $W = W_1 + iW_2$ is the solution of the equation $W_{\overline{z}} = \frac{f_{\overline{z}}}{f}\overline{W}$, then W_1 and W_2 related to by Darboux transformation $W_2 = iW_1[1]$ and $W_1 = -iW_1[1]$.*

2) If W_1 is a solution of the equation $(\triangle - \frac{\triangle f}{f})\psi = 0$, then $W_1 - W_1[1]$ is the solution of the equation $W_{\overline{z}} = \frac{f_{\overline{z}}}{f}\overline{W}$.

3) If W_2 is the solution of the equation $(\triangle + \frac{\triangle f}{f} - 2(\frac{\nabla f}{f})^2)\psi = 0$, then $-iW_2[1] + iW_2$ is a solution of the equation $W_{\overline{z}} = \frac{f_{\overline{z}}}{f}\overline{W}$.

From the Theorem 33 [3] follows, that $W_1 + iW_2 = W$ is the solution of the equation $W_{\overline{z}} = \frac{f_{\overline{z}}}{f}\overline{W}$, then

$$W_2 = f^{-1}\overline{A}[if^2\partial_{\overline{z}}(f^{-1}W_1)] \text{ and } W_1 = -f\overline{A}[i\overline{f}^{2}\partial_{\overline{z}}(fW_2)],$$

where $\overline{A}[\phi] = 2Re\int\phi d\overline{z} = 2Im\int i\phi d\overline{z}$. Therefore,

$$W_2 = -f^{-1}2Im\int f^2\partial_{\overline{z}}(f^{-1}W_1)d\overline{z} \text{ and } W_1 = f2Im\int f^{-2}\partial_{\overline{z}}(fW_2)d\overline{z}.$$

Consider the equation $(\triangle - \frac{\triangle f}{f})\psi = 0$ and take the function f as particular solution of this equation, then by Theorem 33 [3] the function W_1 is the solution of this equation. Consider the Darboux transformation W_1 :

$$W_1 \to W_1[1] = f^{-1}\int\Omega(W_1, f),$$

$$\Omega(W_1, f) = (W_1f_z - W_{1z}f)dz - (W_1f_{\overline{z}} - W_{1\overline{z}}f)d\overline{z} = 2iIm[f^2\partial_{\overline{z}}(f^{-1}W_1)],$$

$$W_1[1] = f^{-1}2iIm\int f^2\partial_{\overline{z}}(f^{-1}W_1)d\overline{z}.$$

Therefore $W_2 = iW_1[1]$.

Now, consider the function $\frac{1}{f}$ as particular solution of the equation $(\triangle - \frac{\triangle f}{f})\psi = 0$, then from Theorem 33 [3] follows, that W_2 is a solution of this equation. Consider the Darboux transformation of W_2:

$$W_2 \to W_2[1] = (\frac{1}{f})^{-1} \int \Omega(W_2, f^{-1}) = f \int \Omega(W_2, f^{-1}),$$

$$\Omega(W_2, f^{-1}) = (W_2 \partial_z (\frac{1}{f}) - W_{2z} \frac{1}{f}) dz - (W_2 \partial_{\overline{z}} (\frac{1}{f}) - W_{2\overline{z}} \frac{1}{f}) d\overline{z} =$$

$$= (-W_2 \frac{f_{\overline{z}}}{f^2} - W_{2z} \frac{1}{f}) dz + \frac{1}{f^2} (W_2 f_{\overline{z}} + W_{2\overline{z}} f) d\overline{z} = 2iIm[f^{-2} \partial_{\overline{z}} (f W_2)].$$

Therefore, $W_1 = -iW_2[1]$.

Remark. In [5] the authors studied intertwining relations, supersymmetry and Darboux transformations for time-dependent generalized Schrodinger equations and obtained intertwiners in an explicit form, it means that it is possible to construct arbitrary-order Darboux transformations for some class of equations. The authors developes a corresponding supersymmetric formulation and proves equivalence of the Darboux transformations with the supersymmetry formalism. In our opinion the method given in this paper it is possible to use in this direction also.

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