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ON ONE NONLINEAR DIFFUSION SYSTEM

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Abstract. One-dimensional nonlinear diffusion system of Maxwell's equations by taking into account Joule's rule and thermal conductivity is considered. Finite difference schemes and splitting-up models are constructed. Graphs of respective numerical experiments are given.

Keywords and phrases: One-dimensional nonlinear Maxwell's system, Joule's rule, thermal conductivity, finite difference schemes, splitting-up models.

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System of Maxwell's equations describing process of penetration of magnetic field into a substance by taking into account Joule's rule and thermal conductivity, has the following form [1]:

$$\frac{\partial \theta}{\partial t} = \nu_m (rot\, H)^2 + div(\kappa\, grad\, \theta), \quad \frac{\partial H}{\partial t} = -rot\ (\nu_m rot\, H), \tag{1}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field, θ is temperature, ν_m and κ are characteristics coefficients of the substance. As a rule these coefficients are functions of argument θ.

Let us consider following initial-boundary value problem for one-dimensional analog of the system (1) with special power-type nonlinearity:

$$\begin{aligned}
&\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left(V^\alpha \frac{\partial U}{\partial x}\right), \quad \frac{\partial V}{\partial t} = V^\alpha \left(\frac{\partial U}{\partial x}\right)^2 + \frac{\partial^2 V}{\partial x^2}, && (x,t) \in \Omega \times (0,T),\\
&U(x,t) = \frac{\partial V(x,t)}{\partial x} = 0, && (x,t) \in \partial\Omega \times (0,T),\\
&U(x,0) = U_0(x), \ \ V(x,0) = V_0(x) \geq Const > 0, && x \in \overline{\Omega},
\end{aligned} \tag{2}$$

where $-1/2 \leq \alpha \leq 1/2$; U_0, V_0 are known functions defined on $\overline{\Omega} = [0,1]$, T is the fixed positive constant.

After introducing new unknown function $V^{1/2} = W$ and notation $2\alpha = \gamma$ the problem (2) can be reduced to the following equivalent form:

$$\begin{aligned}
&\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left(W^\gamma \frac{\partial U}{\partial x}\right), && (x,t) \in \Omega \times (0,T),\\
&\frac{\partial W}{\partial t} = \frac{1}{2} W^{\gamma-1} \left(\frac{\partial U}{\partial x}\right)^2 + \frac{\partial^2 W}{\partial x^2} + \frac{1}{W}\left(\frac{\partial W}{\partial x}\right)^2, && (x,t) \in \Omega \times (0,T),\\
&U(x,t) = \frac{\partial W(x,t)}{\partial x} = 0, && (x,t) \in \partial\Omega \times (0,T),\\
&U(x,0) = U_0(x), \quad W(x,0) = W_0(x) = V_0^{1/2}(x), && x \in \overline{\Omega},
\end{aligned} \tag{3}$$

where $-1 \leq \gamma \leq 1$.

Let us note that same reduction for the problem (2) is also used in [2] as well as for investigation of problem analogical to (2) with first kind of boundary conditions for function V in the work [3].

The questions of existence, uniqueness, regularity of the solutions and numerical resolution of the initial-boundary value problems to the (1), (2) and (3) type models and related parabolic systems are discussed in many works (see, for example, [4]-[20] and references therein).

Note that, system (1) without thermal conductivity can be reduced to integro-differential form [21]. Many responses were followed after publication of the paper [21] (see, for example, [22]-[28]). The questions of existence, uniqueness, asymptotic behavior of the solutions and numerical resolution of some kind of initial-boundary value problems for this type integro-differential models are studied in these works.

It is important to construct and study discrete analogs for the model (1). It is also necessary to investigate this question for one-dimensional analog (2).

Complex nonlinearity dictates also to split the investigated model along the physical process and to investigate basic model by them (see, for example, [2], [3]).

Different type of splitting-up schemes are constructed and investigated for many models of mathematical physics (see, for example, [29]-[31] and references therein).

Investigation of splitting-up schemes along the physical processes for one-dimensional analog of system (1) is the natural beginning of studding this issue. In this direction investigations was made in the works [2], [3] and in a number of other works as well. In the paper [3] semi-discrete additive models for the initial-boundary value problem for system (2) with Dirichlet boundary conditions on temperature is considered. The same question for the second kind of boundary conditions on the function V is considered in [2]. We should note that in [2] the above mentioned splitting-up scheme is constructed for the multidimensional system (1) as well.

The aim of this note is to construct finite-difference schemes for the problem (2). The splitting-up schemes for the problem (3) are considered as well. Let us note that finite difference schemes for (2) type problems are constructed in many works (see, for example, [3], [8], [9], [16], [18] and references therein).

On $[0,1]\times[0,T]$ let us introduce a net with mesh points denoted by $(x_i,t_j)=(ih,j\tau)$, where $i=0,1,...,M$; $j=0,1,...,N$ with $h=1/M$, $\tau=T/N$. The initial line is denoted by $j=0$. The discrete approximation at (x_i,t_j) is designed by u_i^j,v_i^j and the exact solution to the problem (2) by U_i^j, V_i^j.

Let us consider problem (2) with nonhomogeneous right hand sides in both equations of the system and construct the following difference scheme:

$$\frac{u_i^{j+1}-u_i^j}{\tau}=\frac{1}{h^2}\left\{a\left(v_i^{j+1}\right)u_{i-1}^{j+1}\right.$$
$$\left.-\left(a\left(v_i^{j+1}\right)+a\left(v_{i+1}^{j+1}\right)\right)u_i^{j+1}+a\left(v_{i+1}^{j+1}\right)u_{i+1}^{j+1}\right\}+f_i^{j+1}, \tag{4}$$

$$\begin{gathered}\frac{v_i^{j+1}-v_i^j}{\tau}=\sigma a\left(v_i^{j+1}\right)\left(\frac{u_i^{j+1}-u_{i-1}^{j+1}}{h^2}\right)^2+(1-\sigma)\,a\left(v_i^j\right)\left(\frac{u_i^j-u_{i-1}^j}{h}\right)^2\\ +\frac{v_{i+1}^{j+1}-2v_i^{j+1}+v_{i-1}^{j+1}}{h^2}+g_i^{j+1},\end{gathered} \tag{5}$$
$$i=1,2...M-1;\, j=0,1...,N-1,$$

$$u_0^j=u_M^j=v_{x,0}^j=v_{\bar{x},M}^j=0,\quad j=0,1,...,N, \tag{6}$$

$$u_j^0 = U_{0,i} \quad v_j^0 = V_{0,i}, \quad i = 0, 1..., M. \tag{7}$$

Here $a(v) = v^\alpha$, $\alpha \in R$; f, g are given functions and $0 \le \sigma \le 1$ is also a given constant.

The difference scheme (4)-(7) is the first order, i.e. its rate is $O(\tau + h)$.

Theorem 1. *If $-1/2 \le \alpha \le 1/2$, $0 < \sigma \le 1$ and problem (2) has a sufficiently smooth solution, then the difference scheme (4)-(7) is stable, uniquely solvable and its solution converges to the solution of the problem (2) as $\tau \to 0, h \to 0$. The following estimate is also true*

$$\left\| U^i - u^j \right\|_h + \left\| V^j + v^j \right\|_h = O(\tau + h).$$

Here $\|\cdot\|_h$ is a discrete analog of the norm of the space $L_2(0,1)$.

Note that, in the case $\sigma = 0$, for solving the finite difference scheme (4)-(7) at first we solve system (5) by known tridiagonal matrix algorithm and after we solve system (4) by same algorithm, using in both cases suitable boundary and initial conditions from (6), (7). In the case $\sigma \ne 0$, i.e. for solving fully implicit scheme (4)-(7) we must include a method of solving system of nonlinear algebraic equations. So, it is necessary to use Newton iterative process. It is easy to notice that for the solution of the implicit scheme (4)-(7) the numerical algorithms [3] based on a modified Newton method without main changes can be used.

Now, let us consider the splitting-up model for the problem (3). Using the notations:

$$y_t = \frac{y^{j+1} - y^j}{\tau}, \quad \eta_1 + \eta_2 = 1, \quad \eta_1 > 0, \quad \eta_2 > 0,$$
$$y = \eta_1 y_1 + \eta_2 y_2, \quad y_{1t} = \frac{y_1^{j+1} - y^j}{\tau}, \quad y_{2t} = \frac{y_2^{j+1} - y^j}{\tau}$$

let us correspond to the initial-boundary value problem (3) the following additive averaged semi-discrete scheme [2], [3]:

$$\begin{gathered} u_{1t} = \frac{d}{dx}\left(w_1^\gamma \frac{du_1}{dx} \right), \quad \eta_1 w_{1t} = \frac{1}{2} w_1^{\gamma-1} \left(\frac{du_1}{dx} \right)^2, \quad u_{2t} = \frac{d}{dx}\left(w_2^\gamma \frac{du_2}{dx} \right), \\ \eta_2 w_{2t} = \frac{d^2 w_2}{dx^2} + \frac{1}{w_2}\left(\frac{dw_2}{dx} \right)^2, \quad u_1^0 = u_2^0 = U_0, \quad \frac{dw_1^0}{dx} = \frac{dw_2^0}{dx} = W_0. \end{gathered} \tag{8}$$

The following statement takes place [2].

Theorem 2. *If problem (3) has a sufficiently smooth solution, then the solution of scheme (8) converges to the solution of problem (3) as $\tau \to 0$ and the following estimate is true*

$$\left\| U(t_j) - u^j \right\| + \left\| W(t_j) + w^j \right\| = O\left(\tau^{1/2} \right).$$

Here $\|\cdot\|$ is an usual norm of the space $L_2(0,1)$.

As we already mentioned the additive models analogical to (8) for the multi-dimensional system (1) are also constructed and investigated (see, for example, [2]).

At the end we will note that numerical test experiments carried out on the basis on difference schemes (4)-(7) are founded on the above described algorithms (see Fig. 1).

Let us also note that fully discrete analog based on the semi-discrete splitting model (8) is constructed and used. Various numerical experiments are done using these algorithms. These experiments fully agree with theoretical results (see Fig. 2).

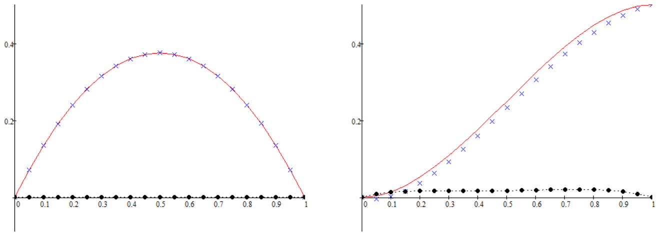

Fig. 1. Exact (solid line) and numerical (marked with $\times$) solutions and differences between exact and numerical solutions (marked with $\bullet$).

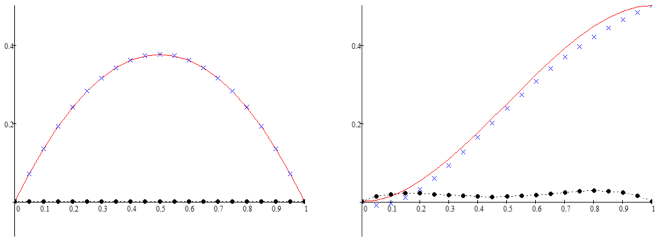

Fig. 2. Exact (solid line) and numerical (marked with $\times$) solutions and differences between exact and numerical solutions (marked with $\bullet$).

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