Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 25, 2011

THE PERIODICITY OF THE SPACE OF GENERALIZED ANALYTIC FUNCTIONS

Buliskeria G.

Abstract. In this paper we consider spaces of generalized analytic functions $\Omega(a, b)$, $a, b \in L_{p,2}$ (see [1],[2]) and show that this spaces as vector spaces on $\mathbb{R}$ have different structures.

Keywords and phrases: Periodicity problem, generating pair, pseudoanalytic functions.

AMS subject classification: 57R45.

Let $F(z), G(z)$ be two complex valued Holder continuous functions defined in some domain such that $Im(\overline{F}G) > 0$. A function $w = \phi F + \psi G$, where ϕ and ψ are real,is called (F, G) pseudoanalytic, if $\phi_{\overline{z}}F + \psi_{\overline{z}}G = 0$. The function $\dot{w} = \phi_z F + \psi_z G$ is called the (F, G) derivative of w. Every generating pair (F, G) has a successor (F_1, G_1) such that (F, G) derivative is (F_1, G_1) pseudoanalytic. The successor is not uniquely determined. A generating pair (F, G) is said to have *minimum period* n if there exists generating pairs (F_i, G_i) such that $(F_0, G_0) = (F, G)$, (F_{i+1}, G_{i+1}) is a successor of (F_i, G_i) and $(F_n, G_n) = (F_0, G_0)$. If such n does not exist, (F, G) is said to have minimum period ∞.

It is known, that w is pseudonalytic iff w satisfies the following Carleman-Bers-Vekua equation

$$w_{\overline{z}} = aw + b\overline{w}, \tag{1}$$

where the function $a(z, \overline{z}), b(z, \overline{z})$ expressed by the generating pair (F, G) by the following identity

$$a = \frac{\overline{G}F_{\overline{z}} - \overline{F}G_{\overline{z}}}{F\overline{G} - \overline{F}G}, b = \frac{FG_{\overline{z}} - GF_{\overline{z}}}{F\overline{G} - \overline{F}G}. \tag{2}$$

Define also the quantities

$$A = \frac{\overline{G}F_z - \overline{F}G_z}{F\overline{G} - \overline{F}G}, B = \frac{FG_z - GF_z}{F\overline{G} - \overline{F}G}. \tag{3}$$

The (F, G)-derivative $\dot{w}$ satisfies the following Carlemnn-Bers-Vekua equation

$$\dot{w}_{\overline{z}} = a\dot{w} - B\overline{\dot{w}}.$$

The functions a, b, A, B are called the characteristic coefficients of the generating pair (F, G).

Proposition 1. [3] *Given functions a, b, A, B are characteristic coefficients of the generating pair if and only if they satisfy the system of differential equations*

$$A_{\overline{z}} = a_z + b\overline{b} - B\overline{B}, \quad B_{\overline{z}} = b_z + (\overline{a} - A)b + (a - \overline{A})B.$$

Proposition 2. [3] *1) The space $\Omega(a,b)$ have period one iff there exist a function A_0 satisfying the equation*

$$A_{0\overline{z}} = a, \quad A_0 - \overline{A}_0 = \overline{a} - a + \frac{1}{b}(b_z + b_{\overline{z}})$$

2)The space $\Omega(a,b)$ have period two iff there exist a functions A_0, A_1, B_0 satisfying the system of equations

$$A_{0\overline{z}} = a_z + b\overline{b} - B_0\overline{B}_0, \quad B_{0\overline{z}} = b_z + (\overline{a} - A_0)b + (a - \overline{A}_0)B_0,$$

$$A_{1\overline{z}} = a_z + b\overline{b} - B_0\overline{B}_0, \quad B_{0z} = b_{\overline{z}} + (A_1 - \overline{a})B_0 + (\overline{A}_1 - a)b.$$

Proposition 3. *Let (F, G) generating pair of (1), then generating pair of the adjoint equation*

$$w_{\overline{z}} = -aw - \overline{B}\overline{w}, \tag{4}$$

is

$$F^* = \frac{2\overline{G}}{F\overline{G} - \overline{F}G}, \quad G^* = \frac{2\overline{F}}{F\overline{G} - \overline{F}G}.$$

We prove, that the characteristic coefficient induced from adjoint generating pair (F^*, G^*) are equal to $-a$ and $-\overline{B}$.

Indeed,

$$\frac{\overline{G}^* F^*_{\overline{z}} - \overline{F}^* G^*_{\overline{z}}}{F^*\overline{G}^* - \overline{F}^* G^*} = \frac{\frac{2F}{\overline{D}}(\frac{2\overline{G}}{D})_{\overline{z}} - \frac{2G}{\overline{D}}(\frac{2\overline{F}}{D})_{\overline{z}}}{\frac{2\overline{G}}{D}\frac{2F}{\overline{D}} - \frac{2G}{\overline{D}}\frac{2\overline{F}}{D}} = \frac{\frac{4F}{\overline{D}}(\frac{\overline{G}_{\overline{z}}}{D} - \frac{\overline{G}}{D^2}D_{\overline{z}}) - \frac{4G}{\overline{D}}(\frac{\overline{F}_{\overline{z}}}{D} - \frac{\overline{F}}{D^2}D_{\overline{z}})}{\frac{4}{D\overline{D}}(F\overline{G} - \overline{F}G)} =$$

$$= \frac{F\overline{G}_{\overline{z}} - G\overline{F}_{\overline{z}}}{D} - \frac{F\overline{G} - \overline{F}G}{D^2}D_{\overline{z}},$$

where $D = F\overline{G} - \overline{F}G, D = -\overline{D}, D_{\overline{z}} = F_{\overline{z}}\overline{G} + F\overline{G}_{\overline{z}} - \overline{F}_{\overline{z}}G - \overline{F}G_{\overline{z}}$. From (1) we have

$$a_1 = \frac{-F_{\overline{z}}\overline{G} - F\overline{G}_{\overline{z}} - \overline{F}_{\overline{z}}G - \overline{F}G_{\overline{z}} + \overline{F}_{\overline{z}}G + \overline{F}G_{\overline{z}}}{D} = -\frac{F_{\overline{z}}\overline{G} - \overline{F}G_{\overline{z}}}{D} \Longrightarrow a = -a_1.$$

Analogically to above

$$b_{1(F^*,G^*)} = \frac{F^*G^*_{G_{\overline{Z}}} - G^*F^*_{\overline{z}}}{F^*\overline{G}^* - \overline{F}^* G^*} = \frac{\frac{G}{D}\overline{G}(\frac{\overline{F}}{D})_{\overline{z}} - \frac{G}{D}\overline{F}(\frac{\overline{G}}{D})_{\overline{z}}}{\frac{G}{\overline{D}}} =$$

$$= \frac{\overline{D}}{D}(\frac{\overline{GF_{\overline{z}}}}{D} - \frac{\overline{GF}}{D^2}D_{\overline{z}} - \frac{\overline{FG_{\overline{z}}}}{D} + \frac{\overline{GF}}{D^2}D_{\overline{z}}) = -\frac{\overline{GF_{\overline{z}}} - \overline{FG_{\overline{z}}}}{D},$$

therefore

$$\overline{b}_1 = -\frac{GF_z - FG_z}{D} \Longrightarrow b_1 = -\overline{B}.$$

By definition (see [1],[2]) the pseudoanalytic functions corresponding to (1) satisfies the following holomorphic disc equation

$$\omega_{\overline{z}} = q(z)\overline{\omega}_z, \quad \text{where} \quad q(z) = \frac{F + iG}{F - iG}.$$

Proposition 4. *Holomorphic disc equation, corresponding to (4) is* $\omega_{\overline{z}} = -\overline{q(z)}\overline{\omega}_z$.
Indeed, coefficient of holomorphic disc equation, corresponding to (4) expressed by the generating pair (F^*, G^*) of (4) as

$$q_1 = \frac{F^* + iG^*}{F^* - iG^*} \Rightarrow q_1 = \frac{\frac{2\overline{G}}{D} + i\frac{2\overline{F}}{D}}{\frac{2\overline{G}}{D} - i\frac{2\overline{F}}{D}} = \frac{\overline{G} + i\overline{F}}{\overline{G} - i\overline{F}} \Rightarrow q_1 = -\overline{q}.$$

Proposition 5. *If equation (1) has the period one, then the equation (4) also has period one.*
The proof immediately follows from the proof of the preceding proposition.

Proposition 6. *The generating pair of the space* $\Omega(a, 0)$ *is* (f, if), *where* $f \neq 0$ *and is solution of the equation* $f_{\overline{z}} = -af$.
Indeed,

$$Im(\overline{f}if) = i|f|^2; (F\overline{G} - \overline{F}G) = f(-i\overline{f}) - \overline{f}(if) = -2i|f|^2,$$

$$a_{(f,if)} = \frac{\overline{f}if_{\overline{z}} - f_{\overline{z}}(-i\overline{f})}{-2i|f|^2} = -\frac{f_{\overline{z}}}{f}, b_{(f,if)} = \frac{fif_{\overline{z}} - f_{\overline{z}}if}{-2i|f|^2} = 0.$$

Consider the particular cases of this theorem. When f is constant, or is complex analytic, we obtain the space of holomorphic functions $\Omega(0, 0)$.

Proposition 7. *If* f *is real and* $f \neq 0$, *then* $(f, \frac{i}{f})$ *generates the space* $\Omega(0, b)$.
The proof obtained from directly computation:

$$Im(f\frac{i}{f}) = 1 > 0, \text{ because } \overline{f} = f; a_{(f,\frac{i}{f})} = \frac{-f(\frac{if_{\overline{z}}}{f^2}) - f_{\overline{z}}(-\frac{i}{f})}{-2i} = 0;$$

$$b_{(f,\frac{i}{f})} = -\frac{-f(\frac{if_{\overline{z}}}{f^2}) - f_{\overline{z}}(\frac{i}{f})}{-2i} = \frac{f_{\overline{z}}}{f}.$$

Proposition 8. *From* $\omega \in \Omega(a, 0)$ *follows, that* $\dot{\omega} \in \Omega(a, 0)$.
By proposition 6 the generating pair of the space $\Omega(a, 0)$ is (f, if). The function $\dot{\omega}$ satisfies the equation (4), therefore it is necessary to compute B from (3). It is easy, that $B = 0$.

In case, when the functions F, G are complex analytic, then from (2) follows, that we obtain the space of holomorphic function $\Omega(0, 0)$, but this space not "isomorphic" to induced from $(1, i)$ generating pair space of holomorphic functions, because at follows from (3), B not equal to zero. From this follows, that this space have period $N > 1$. In [3] shows, that period this space is equal to 2.

Proposition 9. (see [3]) *1)There exist real analytic function* b *in the a neighborhood of the origin, such that the space* $\Omega(0, b)$ *has minimum period infinity.*

2) For each positive integer N *there exists a real analytic function* b *in the neighborhood of the origin, such that the space* $\Omega(0, b)$ *has minimum period* N.

A necessary and sufficient condition that $\Omega(a, b)$ generated by (F, G) have the period one, obtained by Bers, and proved, that this conditions is identity $\frac{F}{G} = \tau(y)$. Bers proved also, that if $\frac{F}{G} = \sigma(x)$, then the minimum period is at most two.

Remark. Markushevich observed that every system of linear partial differential equations

$$c_i u_x + d_i v_x = a_i u_y + b_i v_y, \ i = 1, 2, \tag{5}$$

with sufficiently smooth coefficients $a_1(x, y), ..., d_2(x, y)$ can be written in a form such that $\frac{\partial c_i}{\partial x} = \frac{\partial a_i}{\partial y}, \frac{\partial d_i}{\partial x} = \frac{\partial b_i}{\partial y}$, $i = 1, 2$. In this case the integrals

$$U = \int (a_2 u + b_2 v)dx + (c_2 u + d_2 v)dy, V = \int (a_1 u + b_1 v)dx + (c_1 u + d_1 v)dy \tag{5_1}$$

are path-independent and (U, V) satisfy a system (5_1) which is of the same form as (5) (see [6]). System (5) is said to be embedded into a cycle if there exists a sequence of systems (5), $(5_1), (5_2), ...$ such that (5_i) is related to (5_{i+1}) as (5) was related to (5_1). The cycle is called of finite order n if (5_n) is equivalent to (5), of infinite order if there is no such n. In [4] Lukomskaya (a) proves that every (5) can be embedded into a cycle of infinite order, and (b) gives necessary and sufficient conditions in order that the minimum order n_{min} of a cycle beginning with (5) be 1. In [6] states as an open problem the question on the existence of systems with $n_{min} > 2$. We remark that for elliptic systems the natural setting for this problem is the theory of pseudo-analytic functions [2] and finely result in this direction gives Protter [3] solving the so called *periodicity problem* for pseudoanalytic functions.

Acknowledgements. The authors is grateful to the participants of seminar "Elliptic systems on Riemann surfaces" in Tbilisi State University for many discussions.

R E F E R E N C E S

1. Vekua I.N. Generalized Analytic Functions. *2rd Edition, Nauka*, 1988.

2. Bers L. Theory of Pseudo-analytic Functions. *New York University*, 1953.

3. Protter M.H. The periodicity problem for pseudoanalytic functions. *Ann. Math.*, **2**, 64 (1956), 154-174.

4. Lukomskaya M. On cycles of systems of linear homogeneous differential equations. *Mat. Sbornik (N.S).*, **29** (1951), 551-558.

5. Lukomskaya M. Solution of some systems of partial differential equations by means of inclusion in a cycle. *Akad. Nauk SSSR., Prikl. Mat. Meh.*, **17** (1953), 745-747.

6. Petrovskii M.I.G. On some problems of the theory of the theory of partial differential equations. *Uspekhi Matem. Nauk*, **1** (1946), 44-70.

Received 26.07.2011; revised 22.09.2011; accepted 27.11.2011.

Author's address:

G. Buliskeria
Department of Physics
Iv. Javakhishvili Tbilisi State University
2, Chavchavadze Av., Tbilisi 0128
Georgia
E-mail: buleskiriag@yahoo.com