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THE NEUMANN BVP OF THERMOELASTICITY FOR A TRANSVERSALLY ISOTROPIC PLANE WITH CURVILINEAR CUTS

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Abstract. In the present paper the second (Neumann type) boundary value problem of the theory of thermoelasticity is investigated for a transversally isotropic plane with curvilinear cuts. For solution we used the potential method and constructed the special fundamental matrices, which reduced the problem to a Fredholm integral equations of the second kind.The solvability of a system of singular integral equations is proved by using the potential method and the theory of singular integral equations. For the equation of statics of thermoelasticity we construct one particular solution and we reduce the solution of the second BVP problem of the theory of thermoelasticity to the solution of the second BVP problem for the equation of transversally-isotropic body.

Keywords and phrases: Potential method, integral equation, thermoelasticity.

AMS subject classification: 74G05, 74E10, 74F05.

Introduction. In this present paper the Neumann (second) boundary value problem (BVP) of the theory of thermoelasticity is investigated for a transversally-isotropic plane with curvilinear cuts. The BVP for domains with cuts were studied by a lot of authors by many different methods,for example: The boundary value problems of the theory of elasticity for anisotropic media with cuts were considered in [1,2]. In this paper we intend this result to BVP of the theory of thermoelasticity for the equations transversally-isotropic thermoelastic body.

In the present paper for solution we used the potential method and constructed the special fundamental matrices, which reduced the problem to a Fredholm integral equations of the second kind.The solvability of a system of singular integral equations is proved by using the potential method and the theory of singular integral equations. For the equation of statics of thermoelasticity we construct one particular solution and we reduce the solution of the second BVP problem of the theory of thermoelasticity to the solution of the second BVP problem for the equation of transversally-isotropic body.

Basic equation and BVP. Here we shall be concerned with the plane problem of thermoelasticity (it is assumed that the second component of the three-dimensional displacement vector equals to zero and the other components u_1, u_3 and u_4 depend only on the variables x_1, x_3). In this case the basic two-dimensional equations thermoelasticity for the transversally-isotropic body can be written as follows [3]

$$C(\partial x)u = Bgradu_4, \tag{1}$$

$$\Delta_4 u_4 = a_4 \frac{\partial^2 u_4}{\partial x_1^2} + \frac{\partial^2 u_4}{\partial x_3^2} = 0, \quad j = 0, 1, \tag{2}$$

where
$$C(\partial x)=\|C_{pq}(\partial x)\|_{2x2},\ \ B=\|B_{pq}\|_{2x2},\ \ B_{11}=\beta_1,\ \ B_{22}=\beta',$$
$$B_{12}=B_{21}=0,\ \ C_{11}(\partial x)=c_{11}\frac{\partial^2}{\partial x_1^2}+c_{44}\frac{\partial^2}{\partial x_3^2},\ \ C_{21}(\partial x)=$$
$$C_{12}(\partial x)=(c_{13}+c_{44})\frac{\partial^2}{\partial x_1\partial x_3},\ \ C_{22}(\partial x)=c_{11}\frac{\partial^2}{\partial x_1^2}+c_{44}\frac{\partial^2}{\partial x_3^2},$$

c_{pq} are Hooke's coefficients, $\beta=c_{13}\alpha'+2\alpha(c_{11}-c_{66}),\ \ \beta'=c_{33}\alpha'+2\alpha c_{13},\ a_4=\dfrac{k}{k'},\ \alpha,\alpha'$ are coefficients of temperature extension, k,k' are coefficients of thermal conductivity, $u=(u_1,u_3)$ is a displacement vector, u_4 is the temperature of body.

Let the plane be weakened by curvilinear cuts $l_j=a_jb_j,\quad j=1,2,..,p$. Assume that the cuts $l_j,\quad j=1,...,p$, are simple nonintersecting open Lyapunov's arcs. The direction from a_j to b_j is taken as the positive one on l_j. The normal to l_j will be drawn to the right relative to motion in the positive direction. Denote by D the plane with curvilinear cuts $l_j,\quad j=1,2,..,p,\quad l=\bigcup\limits_{j=1}^{p}l_j$. Let the domain D is filled by homogeneous transversally-isotropic material with the coefficients c_{pq}.

We introduce the notations: $z=x_1+ix_3,\quad \zeta_k=y_1+\alpha_k y_3,\quad \tau_k=t_1+\alpha_k t_3,\quad \sigma_k=z_k-\varsigma_k,\quad z_k=x_1+\alpha_k x_3,\quad \tau=t_1+it_3$.

For equations (1),(2) we pose the following second (Neumann) boundary value problem of static of the theory of thermoelasticity.

Neumann BVP. Find a regular solution of the equation (1),(2) in D, when the stress vector and $\dfrac{\partial_4 u_4}{\partial n}$ are given on both sides of the arcs $l_j,\ \ j=1,2,...,p$. In addition, it is assumed that the principal vector of external force acting on l, stress vector and the rotation at infinity are zero. If we denote by $[Tu]^+([Tu]^-),\quad \left[\frac{\partial_4 u_4}{\partial n}\right]^+\left(\left[\frac{\partial_4 u_4}{\partial n}\right]^-\right)$ the limits on l from the left (right), then the boundary conditions of the problem can be written as follows:
$$[Tu]^+(z)=f^+(z),\quad [Tu]^-(z)=f^-(z),$$
$$\left[\frac{\partial_4 u_4}{\partial n}\right]^+=f_4^+(z),\quad \left[\frac{\partial_4 u_4}{\partial n}\right]^-=f_4^-(z).$$
where $T(\partial x,n)u$ is a stress vector
$$T(\partial x,n)u=\begin{pmatrix} c_{11}n_1\dfrac{\partial}{\partial x_1}+c_{44}n_3\dfrac{\partial}{\partial x_3} & c_{13}n_1\dfrac{\partial}{\partial x_3}+c_{44}n_3\dfrac{\partial}{\partial x_1}\\ c_{44}n_1\dfrac{\partial}{\partial x_1}+c_{13}n_3\dfrac{\partial}{\partial x_1} & c_{44}n_1\dfrac{\partial}{\partial x_1}+c_{33}n_1\dfrac{\partial}{\partial x_3}\end{pmatrix}u,$$
$$\frac{\partial_4}{\partial n}=a_4n_1\frac{\partial}{\partial x_1}+n_3\frac{\partial}{\partial x_3},$$

$f^+,\ f^-,\ f_4^+$, and f_4^- are the known functions on l of the Hölder class H, which have derivatives in the class H^* (for the definitions of the classes H and H^* see[4]) and satisfying at the ends a_j and b_j of l_j, the conditions

$$f^+(a_j) = f^-(a_j), \quad f^+(b_j) = f^-(b_j),$$

$$f_4^+(a_j) = f_4^-(a_j), \quad f_4^+(b_j) = f_4^-(b_j).$$

It is obvious that displacement vector discontinuities along the cut l_j generate a singular stress field in the medium. Hence it is of interest for us to study the solution behavior in the neighborhood of the cuts.

Further we assume that the temperature u_4 is known, when $x \in D$. Substitute the function u_4 in (1) and search the particular solution of the following equation

$$C(\partial x)u = gradu_4.$$

It is easy to prove that $u_0(x)$ is a particular solution of the equation (1)

$$u_0(x) = -\frac{1}{2\pi}\int_D\int \Gamma(x-y)gradu_4(y)dv, \tag{3}$$

where $\Gamma(x-y)$ is the basic fundamental matrix for equation $(C\partial x)u = 0$,

$$\Gamma(x-y) = 2Im\sum_2^3 \|A_{pq}^{(k)}\|_{2x2}\ln\sigma_k,$$

$$A_{11}^{(k)} = \frac{i(-1)^k(c_{44}-c_{33}a_k)}{c_{44}c_{33}(a_2-a_3)}, \quad A_{12}^{(k)} = \frac{(-1)^k(c_{44}+c_{13})}{c_{44}c_{33}(a_2-a_3)}, \quad \alpha_k = i\sqrt{a_k},$$

$$A_{22}^{(k)} = \frac{i(-1)^k(c_{11}-c_{44}a_k)}{c_{44}c_{33}(a_2-a_3)}, \quad \sigma_k = x_1-y_1+\alpha_k(x_3-y_3),$$

$a_k, \quad k = 2,3$ are the positive roots of a characteristic equation

$$c_{44}c_{33}a_k^2 - [c_{11}c_{33}+c_{44}^2-(c_{13}+c_{44})^2]a_k + c_{44}c_{11} = 0.$$

In (3) $gradu_4$ is a continuous vector in D along with its first derivatives and satisfy the following condition at infinity

$$gradu_4 = O(|x|^{-1-\alpha}), \quad \alpha > 0.$$

Thus the general solution of the equation (1) is $u = V + u_0$, where

$$C(\partial x)V = 0, \tag{4}$$

$$[TV]^+) = f^+(z) - [Tu_0]^+ = F^+(z),$$

$$[TV]^- = f^-(z) - [Tu_0]^- = F^-(z).$$

We seek the solution of the problem (4) in the form a single-layer potential

$$V(x) = \frac{1}{\pi}Re\sum_{k=2}^{3}Q_{(k)}\int_l ln(z_k-\zeta_k)[g(y)+ih(y)]d_yS, \tag{5}$$

where g and h are unknown real density vectors, from the Holder class,

$$Q_{(k)} = \begin{pmatrix} q_{11}^{(k)} & q_{21}^{(k)} \\ q_{12}^{(k)} & q_{22}^{(k)} \end{pmatrix} L,$$

$$q_{11}^{(k)} = \frac{(-1)^k(c_{13}+c_{33}a_k)}{c_{33}(a_2-a_3)}, \quad q_{12}^{(k)} = -\frac{(-1)^k(c_{13}a_k+c_{11})}{c_{33}\alpha_k(a_2-a_3)}, \quad \alpha_k = i\sqrt{a_k},$$

$$q_{21}^{(k)} = -\frac{(-1)^k(c_{33}a_k+c_{13})}{c_{33}\alpha_k(a_2-a_3)}, \quad q_{22}^{(k)} = -\frac{(-1)^k(c_{11}+c_{13}a_k)}{a_kc_{33}(a_2-a_3)}, \quad k=2,3,$$

$$L = \frac{c_{33}(\sqrt{a_2}+\sqrt{a_3})}{c_{11}c_{33}-c_{13}^2}\begin{pmatrix} -1 & 0 \\ 0 & -\sqrt{a_2a_3} \end{pmatrix}.$$

From (5), upon acting the operation $T(\partial x, n)$ on the vector V, we get

$$T(\partial x,n)V(x) = \frac{1}{\pi}Re\sum_{k=2}^{3}P_{(k)}\int_l \frac{\partial ln(z_k-\zeta_k)}{\partial s_x}[g(t)+ih(t)]dS, \tag{6}$$

where

$$P_{(k)} = L^{(k)}L, \quad L^{(k)} = (-1)^k\frac{c_{11}c_{33}-c_{13}^2}{c_{33}(a_2-a_3)}\begin{pmatrix} \alpha_k & -1 \\ -1 & \dfrac{1}{\alpha_k} \end{pmatrix}.$$

From (6) to define the unknown density we obtain the following system of singular integral equation of the normal type

$$[T(\partial x,n)V(x)]^{\mp} = \mp g(z) + \frac{1}{\pi}Re\sum_{k=2}^{3}P_{(k)}\int_l \frac{\partial ln(z_k-\zeta_k)}{\partial s_z}[g(y)+ih(y)]d_yS = F^{\pm}(z),$$

$$\frac{\partial}{\partial s} = n_3\frac{\partial}{\partial x_1} - n_1\frac{\partial}{\partial x_3}. \tag{7}$$

From here we deduce that

$$2g(z) = F^+(z) - F^-(z),$$

$$\frac{1}{\pi}Re\sum_{k=2}^{3}iP_{(k)}\int_l \frac{\partial ln(z_k-\zeta_k)}{\partial s_z}h(y)d_yS = \frac{F^+(z)+F^-(z)}{2} \tag{8}$$

$$-\frac{1}{\pi}Re\sum_{k=2}^{3}P_{(k)}\int_l \frac{\partial ln(z_k-\zeta_k)}{\partial s_z}g(y)d_yS = \Omega(z), \quad z\in l.$$

Thus, we have defined the vector g on l. It is not difficult to verify, that $g \in H, g' \in H^*$, and $\Omega \in H, \Omega' \in H^*$. (8) is a system of singular integral equation of normal type with respect to the vector h. We seek the solution of the system (8) in the class h_0 (for the definition of the class h_0 see [4]). Points a_j and b_j are nonsingular ones and the total index in the class h_0 is equal to $-2p$. Let's prove that the adjoint homogeneous equation corresponding to the system (8) has only the trivial solution in the adjoint class.

The adjoint homogeneous system of singular integral equations has the form

$$\frac{1}{\pi}Re\sum_{k=1}^{4}LL_{(k)}\int\limits_{l}\frac{\partial ln(t_k-t_{k0})}{\partial s}\nu(t)ds=0. \tag{9}$$

If the solution of equation (8) in the adjoint class exist, it will satisfy the Hölder's condition on l, vanishing at the points a_j and $b_j, j=1,2,..p$, and having the derivatives in the class H^* ([4]).

Multiplying the system (9) by matrix $a=L^{-1}$, and taking into account the identity $aLL_k=P_{(k)}a$, we obtain

$$\frac{1}{\pi}Re\sum_{k=2}^{3}P_{(k)}\int\limits_{l}\frac{\partial ln(t_k-t_{k0})}{\partial s}a\nu(t)ds=0. \tag{10}$$

Let's assume that (10) has nontrivial solution ν_0 in the adjoint class and construct the potential

$$u_0(z)=\frac{1}{\pi}Re\sum_{k=2}^{3}Q_{(k)}\int\limits_{l}\frac{\partial ln(t_k-z_k)}{\partial s}a\nu_0(t)ds. \tag{11}$$

From (11) we obtain

$$T(\partial z,n)u_0=\frac{\partial\Phi(z)}{\partial s},$$

where

$$\Phi(z)=\frac{1}{\pi}Re\sum_{k=2}^{3}P_{(k)}\int\limits_{l}\frac{\partial ln(t_k-z_k)}{\partial s}a\nu_0(t)ds.$$

By virtue of (9) it is obvious that $\Phi^{\pm}(t_0)=0,\quad t_0\in l$. On the basis of the uniqueness theorem we conclude, that $u_0(t_0)=0$. Then from equality $u_0^+-u_0^-=2const\nu_0$, it follows that $\nu_0=0,\quad t\in l$. Consequently, it follows that the systems (9) and (10) have only the trivial solution.

Thus the homogeneous system corresponding to the system (8) has only $2p$ linearly independent solution. Therefore, the corresponding nonhomogeneous system is solvable in the adjoint class and the solution depends on the 2p arbitrary constants $K_1,K_2,..,K_{2p}$. The choice of these constants stipulates by conditions follows from the single-valuedness of the displacement vector. The displacement vector obtains the increment, while going around l_j, that has to vanish

$$\int\limits_{l}h(t)ds=\frac{c_{33}\sqrt{a_2a_3}-c_{13}}{c_{33}(\sqrt{a_2}+\sqrt{a_3})\sqrt{a_2a_3}}\begin{pmatrix}0 & \sqrt{a_2a_3}\\ -1 & 0\end{pmatrix}\int\limits_{l}[F^-(t)-F^+(t)]ds, \tag{12}$$

Hence (12) is an algebraic equation with respect to unknown constants K_j. Let's prove that the determinant of this system is not zero. In fact, let's take the homogeneous system, corresponding to the conditions

$$F^+ = 0, \quad F^- = 0, \quad (TU)^\infty = 0.$$

Supposing the solution $K_j^{(0)}, j = 1, ..2p$, nontrivial, we construct the potential

$$U_0(z) = \frac{1}{\pi} Re \sum_{k=2}^{3} Q_{(k)} \int_l ln(t_k - z_k) h_0(t) ds, \tag{13}$$

where h_0 is a linear combination of solution $h^{(j)}$. $h_0 = \sum_{k=1}^{4p} K_j^{(0)} h^{(j)}$, and $h^{(j)}$ are linearly independent solutions of the homogeneous equation corresponding to (7). $h^{(j)}$ have to satisfy the following condition $\int_{l_j} h_{(0)} ds = 0, j = 1, .., p$.

Then the potential (13) is regular at infinity and by the uniqueness theorem $U_0 = 0$. But we have the following equality

$$\left(\frac{\partial U_0}{\partial s}\right)^+ - \left(\frac{\partial U_0}{\partial s}\right)^- = Lh^{(0)} = 0.$$

Hence we conclude that $K_j = 0$, which contradicts the assumption. Thus the solvability of the problem is proved.

Repeating word by word the above reasoning we can show that

$$u_4 = Im \frac{1}{\pi i \sqrt{a_4}} \int_l \ln(z_4 - \varsigma_4)(g_4 + ih_4) d_y S,$$

where $z_4 = x_1 + i\sqrt{a_4}x_3$, $\varsigma_4 = y_1 + i\sqrt{a_4}y_3$, $2g_4(y) = f_4^+(y) - f_4^-(y)$, and h_4 is a solution of the following integral equation

$$\frac{1}{\pi} Re \int_l \frac{\partial}{\partial s} \ln(z_4 - \varsigma_4) h_4(y) d_y S = \frac{f_4^= + f_4^-}{2} - \frac{1}{\pi} Im \int_l \frac{\partial}{\partial s} \ln(z_4 - \varsigma_4) g_4(y) d_y S.$$

Here we assume that $\int_l f_4^\pm ds = 0$.

Let us consider a particular case, when the plane has only one rectilinear cut ab along the real axis. Assuming that the principal vector of external forces vanishes at infinity. Then the stress vector outside of the segment ab is calculated by the formula

$$TU_0(z) = \frac{1}{2\pi} Re \sum_{k=2}^{3} P_{(k)} \int_l \left[\frac{f^+(t) - f^-(t)}{t - z_k} dt + \frac{1}{X(z_k)} \int_l \frac{X^+(t)(f^+(t) + f^-(t))}{t - z_k} dt \right],$$

$$\frac{\partial u_4^{(0)}(z)}{\partial n} = \frac{1}{2\pi} Re \int_l \left[\frac{f_4^+(t) - f_4^-(t)}{t - z_4} dt + \frac{1}{X(z_4)} \int_l \frac{X^+(t)(f_4^+(t) + f_4^-(t))}{t - z_4} dt \right],$$

where $X(z_k) = \sqrt{z_k - a)(b - z_k)}$, $\quad k = 2, 3, 4$ is a holomorphic function on the plane cut along the arc ab and $X^+(t) = \sqrt{t - a)(b - t)}$.

R E F E R E N C E S

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