

ON THE GALOIS GROUP SOME FUCHSIAN SYSTEMS

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Abstract. The aim of this paper is to give one new results of the differential Galois theory of linear ordinary differential equations. In particular, we compute differential Galois group for special type non-resonant Fuchsian system.

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The basic fact and definitions. The Galois theory of linear ordinary differential equations was created by Picard and Vessiot at the end of the nineteenth century and it is a theory analogous to the classical Galois theory of polynomials. It is known, that if given the differential field K of coefficients and the extension field of L of K generated by the solutions, then the essential information about the solutions is contained in a group of hidden symmetries of the equation: the Galois group. This group is a linear algebraic group and, as in the classical theory, a correspondence between subfields and subgroups is satisfied (see [1]).

The first rigorous proofs of nonsolvability of differential equations in a finite form (in terms of quadratures and elementary functions) were known before the middle of the XIXth century. The Liouville theory does not imply that a "simple equation" necessarily has a "simple solution". For example, the Bessel equation

$$y'' + xy' + (x^2 - \nu^2)y = 0,$$

which is integrable in a finite form for $\nu = \frac{2n+1}{2}$, $n \in \mathbb{Z}$, has the solution

$$J_{n+\frac{1}{2}}(x) = (-1)^{n+\frac{1}{2}} \frac{1}{\sqrt{\pi}} \frac{d^n}{d(x^2)^n} \frac{\sin x}{x},$$

where $n = 0, 1, 2, \dots$. In addition to the group analysis mentioned above, the differential operators factorization method, applied in combination with Kummer-Liouville and Darboux transformations, is also an effective way of integrating ODE.

Let $D = \{a_1, \dots, a_n\}$ be a finite set of points on the Riemann sphere CP^1 , and let z be a parameter on CP^1 . It is assumed that $z = \infty$ is not among the points of D . Consider Fuchsian systems of first-order linear differential equations with the set of singularities of D , i.e., systems of linear differential equations with first-order poles at points from D , which have the form

$$df = \omega f, \tag{1}$$

where f is a column vector of p components, and the coefficient matrix ω has the form

$$\omega = \sum_{i=1}^n \frac{B_i}{z - a_i} dz$$

with constant $p \times p$ complex matrices B_i satisfying the condition $\sum_{i=1}^n B_i = 0$. Let

$$\chi : \pi_1(CP^1 - D, z_0) \rightarrow GL(n, C)$$

be the representation of the fundamental group. The Riemann-Hilbert problem formulated as follows: realize the representation χ as the monodromy representation of some Fuchsian system of the form (1) (see [3]).

According to Lappo-Danilevsky it is possible to express analytically coefficients of a Fuchs type system by the monodromy matrices, provided these matrices satisfy certain conditions. Lappo-Danilevsky showed that if the monodromy matrices M_1, \dots, M_m are close to $\mathbf{1}$, then coefficients A_j of the system of differential equations of the Fuchs type $\frac{df}{dz} = \left(\sum_{j=1}^m \frac{A_j}{z-s_j} \right) f$ are expressed by the singular points s_j and monodromy matrices M_j via noncommutative power series $A_j = \frac{1}{2\pi i} \tilde{M}_j + \sum_{1 \leq k, l \leq n} \xi_{kl}(s) \tilde{M}_k \tilde{M}_l + \dots$, where ξ_{kl} is a function depending on the singular points which can be given explicitly from $s \in S$, and $\tilde{M}_j = M_j - \mathbf{1}$. Algebraic version of the Riemann-Hilbert monodromy problem is known in the differential Galois theory under the name of *inverse problem* [1].

The problem is formulated as follows. Let k be a differential field with the field of constants C and $D(y) = y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y$ be a differential operator with coefficients in k . To the operator D one assigns the so-called *Picard-Vessiot field* K , whose automorphism group G is the Galois group of the equation $D(y) = 0$, isomorphic to some subgroup of $GL_n(C)$. More precisely the inverse problem is formulated as follows:

Given a group G , find an extension of k with Galois group G .

Constructive character of this problem will become clear if one recalls that a system of linear differential equations is solvable in the class of Liouville functions if and only if the identity component of the Galois group of the equation is a solvable group (see [5]).

Differential Galois theory is also a basis for establishing integrability of a function in the class of elementary functions, and after the elegant works by Kovacic, Davenport and Singer the algorithmic approach became an alternative to the Slagle's heuristic approach. Similar ideas appear in the theory of integrable systems (see [5]).

Relationship between the Galois group of the regular system and its monodromy group is expressed by the following result:

The differential Galois group of the regular system $df = \omega f$ is the Zariski closure of its monodromy group (see [5]).

We begin with some basic definitions.

Definition.

1) A *differential ring* (R, Δ) is a ring R with a set $\Delta = \{\partial_1, \dots, \partial_m\}$ of maps (*derivations*) $\partial_i : R \rightarrow R$, such that

1. $\partial_i(a + b) = \partial_i(a) + \partial_i(b)$, $\partial_i(ab) = \partial_i(a)b + a\partial_i(b)$ for all $a, b \in R$, and
2. $\partial_i \partial_j = \partial_j \partial_i$ for all i, j .

2) The ring $C_R = \{c \in R \mid \partial(c) = 0 \forall \partial \in \Delta\}$ is called the *ring of constants of R* .

When $m = 1$, we say R is an *ordinary differential ring* (R, ∂) . We frequently use the notation a' to denote $\partial(a)$ for $a \in R$. A differential ring that is also a field is called a *differential field*. If k is a differential field, then C_k is also a field.

Example.

- 1) $(C^\infty(R^m), \Delta = \{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\})$ = infinitely differentiable functions on R^m .
- 2) $(C(x_1, \dots, x_m), \Delta = \{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\})$ = field of rational functions.
- 3) $(C[[x]], \frac{\partial}{\partial x})$ = ring of formal power series $C((x))$ = quotient field of $C[[x]] = C[[x]][\frac{1}{x}]$
- 4) $(C\{\{x\}\}, \frac{\partial}{\partial x})$ = ring of germs of convergent series $C(\{x\})$ = quotient field of $C\{\{x\}\} = C\{\{x\}\}[\frac{1}{x}]$
- 5) $(M_O, \Delta = \{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\})$ = field of functions meromorphic on open, connected $0 \subset C^m$.

The following result (see [5]) shows, that many examples reduce to Example 5) above:

Theorem 1. *Any differential field k , finitely generated over \mathbb{Q} , is isomorphic to a differential subfield of some M_O .*

We wish to consider and compare three different versions of the notion of a linear differential equation.

Proposition. *Let (k, ∂) be a differential field. Then the following three notions are equivalent*

1. *scalar linear differential equation is an equation of the form*

$$L(y) = a_n y^{(n)} + \dots + a_0 y = 0, \quad a_i \in k.$$

2. *A matrix linear differential equation is an equation of the form*

$$Y' = AY, \quad A \in GL_n(k),$$

where $GL_n(k)$ denotes the ring of $n \times n$ matrices with entries in k .

3. *A differential module of dimension n is an n -dimensional k -vector space M with a map $\partial : M \rightarrow M$ satisfying*

$$\partial(fm) = f'm + f\partial m \text{ for all } f \in k, m \in M.$$

Let $L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_0y = 0$. If we let $y_1 = y, y_2 = y', \dots, y_n = y^{(n-1)}$, then we have

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix}.$$

We shall write this last equation as $Y' = A_L Y$ and refer A_L as the companion matrix of the scalar equation and the matrix equation as the companion equation. Clearly

any solution of the scalar equation yields a solution of the companion equation and *vice versa*.

Given $Y' = AY$, $A \in GL_n(k)$, we construct a differential module in the following way: Let $M = k^n$, e_1, \dots, e_n the usual basis. Define $\partial e_i = -\sum_j a_{j,i} e_j$, i.e., $\partial e = -A^t e$. Note that if $m = \sum_i f_i e_i$ then $\partial m = \sum_i (f_i' - \sum_j a_{i,j} f_j) e_i$. In particular, we have that $\partial m = 0$ if and only if

$$\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}' = A \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}.$$

It is this latter fact that motivates the seemingly strange definition of this differential module, which we denote by (M_A, ∂) .

Conversely, given a differential module (M, ∂) , *select* a basis $e = (e_1, \dots, e_n)$. Define $A_M \in GL_n(k)$ by $\partial e_i = \sum_j a_{j,i} e_j$. This yields a matrix equation $Y' = AY$. If $\bar{e} = (\bar{e}_1, \dots, \bar{e}_n)$ is another basis, we get another equation $Y' = \bar{A}Y$. If f and \bar{f} are vectors with respect to these two bases and $f = B\bar{f}$, $B \in GL_n(k)$, then

$$\bar{A} = B^{-1}AB - B^{-1}B'.$$

Theorem 2. *Let the system (1) is non-resonant. Then the differential Galois group is generated by $e^{2\pi i B_j}$, $j = 1, \dots, m$.*

First we remark, that in non-resonant case the differential Galois group of equation coincides with closure of the subgroup of $GL_n(C)$ generated by monodromy matrices of the system. Next, use expression

$$\Phi_j(\tilde{z}) = U_j(z)(z - s_j)^{\Lambda_j}(\tilde{z} - s_j)^{E_j},$$

of the solution to system (1) and the fact, that a) the sums $\rho_j^i + \varphi_j^i$ are the eigenvalues of the matrix-residue and b) equal eigenvalues of E_j occupy consecutive position on the diagonal and that the matrix E_j is block-diagonal, with diagonal blocks of sizes equal to their multiplicities. Here \tilde{z} denotes the coordinates on the universal covering, U_j is holomorphic invertible at s_j , $\Lambda_j = \text{diag}(\alpha_j^1, \dots, \alpha_j^n)$, $\alpha_j^1 \geq \dots \geq \alpha_j^n$, is the diagonal matrix, with integer entries and E_j are upper triangular matrices. Hence, if the eigenvalues of B_j are non-resonant, then to equal eigenvalues of E_j there correspond equal eigenvalues of A_j , the matrices A_j and E_j commute and $B_j = U_j(0)(A_j + E_j)U_j(0)^{-1}$ (see [2],[3],[4]). One has $M_j = G_j^{-1}e^{2\pi i E_j}G_j$. From this for the Jordan normal form $JNF(M_j)$ of M_j one has $JNF(M_j) = JNF(E_j) = JNF(A_j + E_j) = JNF(B_j)$. The Theorem 2 is proved.

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R E F E R E N C E S

1. Magid R. Differential Galois theory. *Notices Amer. Math. Soc.*, **46**, 9 (1999), 1041-1049.
2. Kohno M. Global Analysis in Linear Differential Equations. Mathematics and its Applications, 471. Kluwer Academic Publishers, Dordrecht, 1999.
3. Bolibrukh A.A. Hilberts twenty-first problem for linear Fuchsian systems. *Proc. Steklov Inst. Math.*, **5**, 1995.
4. Giorgadze G. Regular systems on Riemann surfaces. *Journal of Math. Sci.*, **118**, 5 (2003), 5347-5399.
5. Van der Put M., Singer M. Galois Theory of Linear Differential Equations. Grundlehren der Mathematischen Wissenschaften, 328. *Springer-Verlag, Berlin*, 2003.

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