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ON ONE AVERAGED INTEGRO-DIFFERENTIAL MODEL

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Abstract. One nonlinear integro-differential equation is considered. The equation arises at describing penetration of a magnetic field into a substance and is based on the well known Maxwell system. Large time behavior of solution of the initial-boundary value problem is studied. Corresponding finite difference scheme is considered as well. Results of numerical experiments are given.

Keywords and phrases: Nonlinear integro-differential equation, asymptotic behavior, finite difference scheme.

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On mathematical simulation of the process of penetration of a magnetic field into a substance the following type of nonlinear integro-differential model arises

$$\frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\left[a\left(\int_o^t \left(\frac{\partial U}{\partial x}\right)^2 d\tau \right) \frac{\partial U}{\partial x} \right] = 0, \tag{1}$$

where $a = a(S) \geq a_0 = Const > 0$ is a given function of its argument.

The models of type (1) at first were introduced in [1] where reduction of well known nonlinear Maxwell system [2] to the integro-differential form was made. In [3] some generalization of (1) type equations is given. One-dimensional simple analog, called by averaged integro-differential model by author, describing the same physical process has the following form

$$\frac{\partial U}{\partial t} - a\left(\int_0^t \int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx d\tau \right) \frac{\partial^2 U}{\partial x^2} = 0. \tag{2}$$

Many works are dedicated to the investigation and numerical resolution of (1) and (2) type models. Especially, in [1], [3]-[10] solvability and uniqueness of the initial-boundary value problems are studied. Asymptotic behavior of solutions as $t \to \infty$ is investigated in many works also. (see, for example, [8],[10]-[19] and references therein). Numerical resolution by finite difference scheme is given in works [10], [12], [15]-[18], [20], [21] and in a number of other works as well.

The aim of this note is to study asymptotic behavior of solution as $t \to \infty$ and to construct approximate solutions for one generalization of the equation (2) by adding monotonic nonlinear term. This equation has the form

$$\frac{\partial U}{\partial t} - a\left(\int_0^t \int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx d\tau \right) \frac{\partial^2 U}{\partial x^2} + |U|^{q-2}U = 0, \tag{3}$$

where $q \geq 2$.

Let us note that such kind generalization for the equation (1) is made and discussed in the work [21].

In the $[0,1] \times [0,\infty)$ let us consider the following initial-boundary value problem

$$\begin{gathered} U(0,t) = U(1,t) = 0, \\ U(x,0) = U_0(x), \end{gathered} \tag{4}$$

where $U_0 = U_0(x)$ is a given function.

It is not difficult to get the following statement.

Theorem 1. *If $a(S) \geq a_0 = Const > 0$, $q \geq 2$, $U_0 \in L_2(0,1)$ then problem (3),(4) has not more than one solution and the following asymptotic property takes place*

$$\|U(x,t)\| \leq Ce^{-a_0 t}.$$

Here $\|\cdot\|$ is the usual norm of the space $L_2(0,1)$.

In the rectangle $[0,1]\times[0,T]$ let us introduce uniform grid with mesh points denoted by $(x_i,t_j) = (ih, j\tau)$, where $i = 0,1,...,M$; $j = 0,1,...,N$ with $h = 1/M$, $\tau = T/N$. The discrete approximation at (x_i,t_j) is designed by u_i^j and the exact solution to the problem (3),(4) by U_i^j.

Using usual notations and the methods of construction of difference schemes (see, for example, [22]) let us construct following finite difference scheme for problem (3),(4):

$$\begin{gathered} \frac{u_i^{j+1} - u_i^j}{\tau} - a\left(\tau h \sum_{k=1}^{j+1}\sum_{l=1}^{M} \left(u_{\bar{x},l}^k\right)^2\right) u_{\bar{x}x,i}^{j+1} + |u_i^{j+1}|^{q-2} u_i^{j+1} = 0, \\ i = 1,2,...M-1; \quad j = 0,1...N-1, \end{gathered} \tag{5}$$

$$u_0^j = u_M^j = 0, \quad j = 0,1...,N, \tag{6}$$

$$u_i^0 = U_{0,i}, \quad i = 0,1...,M. \tag{7}$$

The following statement takes place.

Theorem 2. *If $a(S) = 1+S$, $q \geq 2$ and the initial-boundary value problem (3),(4) has the sufficiently smooth solution $U = U(x,t)$ then the finite difference scheme (5)-(7) converges and the following estimate is true*

$$\left\|u^j - U^j\right\|_h \leq C(\tau + h).$$

Here $\|\cdot\|_h$ is a discrete analog of the norm of the space $L_2(0,1)$ and C is a positive constant independent of τ and h.

Note that for solving the finite difference scheme (5)-(7) we use a method of solving system of nonlinear algebraic equations. So, it is necessary to use Newton iterative process [23]. According to this method the great numbers of numerical experiments are carried out. These experiments agree with theoretical investigations.

The test given on the figures below has the form $U(x,t) = x(1-x)\cos(x+t)$ (see, Fig. 1).

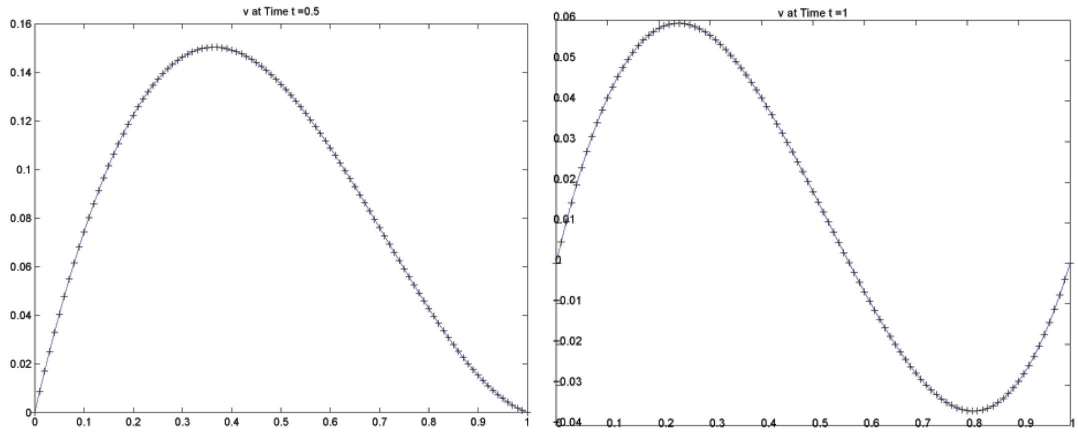

Fig. 1: The solutions at t = 0.5 and t=1. The exact solution is solid line and the numerical solution is marked by *.

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