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CATEGORY OF A-GROUPS OVER A RING A

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Abstract. The notion of an A-group over a ring A is introduced in three different ways. The key idea consists in realizing a tensor completion of an A-group in the form of a concrete structure using free products with union. As a result, the description of free A-groups and free A-products is obtained in terms of free group structures.

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Introduction. In the present paper, for an arbitrary associative ring with unity we define a new category of A-groups in three different ways. In Lyndon [1], the general notion of an A-group was introduced and some results on free A-groups were obtained. For a limited class of (so-called binomial) rings, in his famous work [2] F. Hall formulated for the first time the axiomatic notion of a nilpotent A-group which turned out very productive in the general theory of nilpotent groups. In [3], A. Myasnikov and V. Remeslennikov refined Lyndon's definition of an A-group by introducing one more additional axiom, according to which all abelian subgroups of an A-group are ordinary A-modules. This refinement is a natural generalization of an A-module to the non-commutative case. In [3], the basic notions of the theory of A-groups are given in refined form and the tensor completion structure which is the key structure in the category of A groups is defined. The tensor completion is used in this paper in defining free structures in the category of A-groups, including the notion of an A-free group.

1. Basic definitions and examples.

1.1. Let A denote an arbitrary associative ring with unity, and G a group. Let us enrich the group language $L_{gr} = \langle \cdot\, , ^{-1}, e\rangle$ as follows: $L_{gr} \cup \{f_\alpha(x) \mid \alpha \in A\}$ where $f_\alpha(x)$ is a unary operation denoted by $f_\alpha(g) = g^\alpha \;\forall\, g \in G$.

Definition 1. The set G will be called a **Lyndon A-group** if on it the operations $\cdot$, $^{-1}$, e, $f_\alpha(x)$ are defined and the following axioms are fulfilled

I. Group axioms;

II. $g^1 = g$, $g^0 = e$, $e^\alpha = e$,
$g^{\alpha+\beta} = g^\alpha \cdot g^\beta$, $g^{\alpha\beta} = (g^\alpha)^\beta$,
$(h^{-1}gh)^\alpha = h^{-1}g^\alpha h$
for any α, $\beta \in A$ and g, $h \in G$.

Since Axioms I and II are identities, we can speak of a veriety of A-groups, A-isomorphisms, A-homomorphisms, free A-groups. Let $\mathfrak{L}_A$ denote the category of all Lyndon A-groups with A-homomorphisms.

Definition 2. Let G, $H \in \mathfrak{L}_A$. Then a homomorphism $\varphi : G \to H$ is called an A-**homomorphism** if $(g^\alpha)^\varphi = (g^\varphi)^\alpha$ for any $g \in G$, $\alpha \in A$.

1.2. An A-torsion-free group

Definition 3. An element g of a group $G \in \mathfrak{L}_A$ is **called periodic** if $g^\alpha = e$ for some $0 \neq \alpha \in A$. A group G that does not contain non-unit periodic elements is **called an A-torsion-free group**.

Proposition 1. *The set $O(g) = \{\alpha \in A |\ g^\alpha = e\}$ is the right ideal in a ring A (the **ordinal ideal of an element** g).*

Let us consider a set of axioms(quasi-identities) for groups from the class $\mathfrak{L}_A$:

$$\forall\, e \neq g \in G \;\; g^\alpha = e \longrightarrow g = e.$$

As easily seen, all A-torsion-free A-groups form a quasi-variety of A-groups.

1.3 A-group categories. In [3], A. Myasnikov and V. Remeslennikov introduced a new category of A-groups by adding one more axiom

$$(\mathrm{MR}): \quad \forall\, g, h \in G \;\; [g, h] = 1 \longrightarrow (gh)^\alpha = g^\alpha h^\alpha.$$

It is obvious that all A-modules over a ring A satisfy the (MR) axiom. An example given in [3] shows that $\mathfrak{M}_A$ is a proper subclass in $\mathfrak{L}_A$.

Examples. Most of natural examples of A-groups lie in the class $\mathfrak{M}_A$:

1) An arbitrary group is a $\mathbb{Z}$-group;
2) an abelian divisible group from $L_{\mathbb{Q}}$ is a $\mathbb{Q}$-group;
3) a group of the period m is a $\mathbb{Z}/m\mathbb{Z}$-group;
4) an arbitrary A-operator group from $\mathfrak{L}_A$ with a ring of operators A is an A-group;
5) a module over a ring A is an abelian A-group;
6) a free A-group from $\mathfrak{L}_A$ is an $\mathfrak{L}_A$-group from $\mathfrak{A}$;
7) an arbitrary nilpotent A group over a binomial ring A, which was introduced by F. Hall in [2], is an A-group from $\mathfrak{M}_A$ (see Subsect. 2.2);
8) an arbitrary pro-p-group is a $\mathbb{Z}_{p^\infty}$-group over a ring of integer p-adic numbers $\mathbb{Z}_{p^\infty}$;
9) an arbitrary pro-finite group is a $\widehat{\mathbb{Z}}$-group, where $\widehat{\mathbb{Z}}$ is the total completion $\mathbb{Z}$ in pro-finite topology;
10) a complex (real) unipotent Lie group is a $\mathbb{G}$-group ($\mathbb{R}$-group).

2. Nilpotent A-groups

2.1 Let $c > 1$ be a natural number. Denote by $\mathfrak{N}_{c,A}$ the category of nilpotent A-groups of nilpotence step c from the class $\mathfrak{L}_A$, i.e. of those A-groups for which the identity

$$\forall\, x_1, \ldots, x_{c+1} \;\; [x_1, \ldots, x_{c+1}] = 1$$

is fulfilled.

Denote by $\mathfrak{N}^{\circ}_{c,A}$ the category of nilpotent groups of step c, for which the (MR) axiom holds true. The structure of A-groups without the (MR) axiom is very complicated and that is why only A-groups with the property (MR) are investigated in most papers. In what follows we will consider only A-groups with this axiom.

2.2. Hall nilpotent groups. In order to introduce this notion we have to restrict the class of considered rings.

Definition 4. A ring A is **called binomial** if A is an integrity domain with $\mathbb{Z}$ as a subring and contains, along with every element α, all binomial coefficients

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}, \quad n \in \mathbb{Z}.$$

Definition 5. A nilpotent group G of nilpotence step c is **called a Hall A-group** if for any x from G and α from A, an element $x^{\alpha} \in G$ is defined uniquely and the following axioms are fulfilled ($x, y, x_1, \ldots, x_n$ are arbitrary elements from G; α, β are arbitrary elements from A):

1) $x^1 = x$, $x^{\alpha+\beta} = x^{\alpha}x^{\beta}$, $x^{\alpha\beta} = (x^{\alpha})^{\beta}$;

2) $(y^{-1}xy)^{\alpha} = y^{-1}x^{\alpha}y$;

3) $x_1^{\alpha}x_2^{\alpha}\cdots x_n^{\alpha} = (x_1, x_2 \ldots, x_n)^{\alpha}\tau_2(x)^{\binom{\alpha}{2}}\cdots\tau_c(x)^{\binom{\alpha}{c}}$,
where $\tau_k(x)$ is the k-th Petrescu word of $x_1, x_2 \ldots, x_n$.

Let us describe Petrescu words in more detail. Assume that $x_1, x_2 \ldots, x_n$ is the base of a free group F. For each natural k the k-th Petrescu word $\tau_k(x_1, x_2 \ldots, x_n) = \tau_k(x)$ is defined recurrently from the relation

$$x_1^k x_2^k \cdots x_n^k = \tau_1(x)^k \tau_2(x)^{\binom{k}{2}} \cdots \tau_{k-1}(x)^{\binom{k}{k-1}}\tau_k(x).$$

In particular, $\tau_1(x) = x_1x_2\cdots x_n$; $\tau_2(x) = \prod\limits_{i>j}[x_i, x_j] \pmod{\gamma_3(F)}$.

Denote the category of Hall A-groups by $\mathfrak{H}\mathfrak{N}_{c,A}$. We are going to show that the structure of groups from $\mathfrak{N}_{c,A}$ much differs from the structure of groups from $\mathfrak{H}\mathfrak{N}_{c,A}$. To this end, following [4] we recall the structure of a free A-group in the variety $\mathfrak{H}\mathfrak{N}_{2,A}$. Our consideration is confined to two binomial rings $A = \mathbb{Q}[t]$, $A = \mathbb{Q}(t)$, where $\mathbb{Q}$ is the field of rational numbers.

Denote by G_0 a free 2-step nilpotent A-group in the category $\mathfrak{H}\mathfrak{N}_{2,A}$ with generators x, y. It is well known that the Maltsev base of this group consists of three elements x, y, $[y, x]$. The general form of an element $g \in G_0$ is

$$g = x^{\gamma}y^{\delta}[y, x]^{\varepsilon}, \quad \gamma, \delta, \varepsilon \in A.$$

In particular, in this group the commutant G_0' is a free A-module of rank 1 with generator $[y, x]$.

If now G is a free A-group in the variety $\mathfrak{N}_{2,A}^0$, then in [4] it is shown that G' is a free A-module of infinite rank and the base of this module is found.

3. Free products of A- groups. The tensor completion, which is the basic operation in the class of A-groups $\mathfrak{M}_A$, is investigated in the paper [3]. For modules, it generalizes in a natural manner the notion of expansion of a ring of scalars to the non-commutative case [5]. The ideas of this generalization for the class of nilpotent groups are stated in [6]. Here the tensor completion is used in defining free structures in the category of A-groups [7], including the notion of an A-free group. For the completeness of our discussion, we recall here the definition of a tensor completion.

Definition 6. Let G be an A-group, $\mu : A \to B$ be a ring homomorphism. Then a B-group G^B is called a **tensor B-completion** of the A-group G if G^B satisfies the following universal property:

1) there exists an A-homomorphism $\lambda : G \to G^B$ such that $\lambda(G)$ B-generates G^B, i.e. $\langle \lambda(G) \rangle_B = G^B$;

2) for any B-group H and A-homomorphism $\varphi : G \to H$ which is compatible with μ (i.e. such that $(g^\alpha)^\varphi = (g^\varphi)^{\mu(\alpha)}$), there exists a B-homomorphism $\psi : G^B \to H$ that makes the following diagram commutative:

$$\begin{tikzcd} G \arrow[r, "\lambda"] \arrow[d, "\varphi"'] & G^B \arrow[dl, dashed, "\psi"] \\ H & \end{tikzcd}, \quad \lambda\psi = \varphi.$$

Note that if G is an abelian A-group, then $G^B \cong G \underset{A}{\otimes} B$ is the tensor product of a A-module G by a ring B. In [3], it is proved that for any A-group G and any homomorphism $\mu : A \to B$, the tensor completion G^B exists always and it is unique to within an isomorphism.

Let us formulate the notion of a free A-group. Assume that A is an associative ring with unity, X is an arbitrary variety.

Definition 7. A A-group $F_A(X)$ with a set of A-generators X is called a free A-group with base X if for each A-group G an arbitrary mapping $\varphi_0 : X \to G$ continues to an A-homomorphism $\varphi : F_A(X) \to G$. A set X is called a **set of free A-generators** $F_A(X)$. The power $|X|$ is called the **rank of the group $F_A(X)$**.

Theorem 1. *For any X and A, a free A-group $F_A(X)$ exists in the class $\mathfrak{M}_A$ and it is unique to within an A-isomorphism.*

Definition 8. Let G_i, $i \in I$, be A-groups. An A-group $\underset{A}{*} G_i$ is called a **free product** in the category $\mathfrak{M}_A$ if A-homomorphisms $\varphi_i : G_i \to \underset{A}{*} G_i$ are such that for any A-homomorphisms $\psi_i : G_i \to H$, where H is an arbitrary A-group, there exists an

A-homomorphism $\psi : \underset{A}{*} G_i \mapsto H$ that makes the following diagrams commutative:

$$\begin{tikzcd} G_i \arrow[r, "\varphi_i"] \arrow[d, "\psi_i"'] & \underset{A}{*} G_i \arrow[dl, "\psi"] \\ H & \end{tikzcd} \qquad (i \in I)$$

and the group $\underset{A}{*} G_i$ is A-generated by the set $\{\varphi_i(g_i) |\ g_i \in G_i,\ i \in I\}$.

From the category argument it follows that the group $\underset{A}{*} G_i$ is defined uniquely to within an A-isomorphism.

Theorem 2. *Let A be a ring containing $\mathbb{Z}$ as a subring, G_i, $i \in I$ be some set of A-groups. Then $\underset{A}{*} G_i \cong (*G_i)^A$.*

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