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# EULER CHARACTERISTICS OF THE INTERSECTION OF STABLE QUADRATIC MAPPINGS

Aliashvili T.

**Abstract**. The topology of the fibers of real quadratic mappings was investigated by many authors. In particular, intersections of two and three real quadrics have been studied in detail; the most complete results were obtained for nonsingular (smooth) intersections (see [1]). In this paper, we concentrate on the singular fibres of stable quadratic mappings; stability is understood in the sense of the singularity theory of smooth mappings [2]. For completeness, we present a few related concepts and definitions.

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As is known, in many problems of variational calculus and optimal control theory, an important role is played by the topological structure of the fibers of a given quadratic mapping of Euclidean spaces [1]. Similar problems arise in algebraic geometry and singularity theory [2]; they emerge also in the theory of mechanisms and robotics in studying the configuration spaces of hinge mechanisms and kinematical robots [3]. Moreover, they are closely related to construction of algebraic models for smooth manifolds (see [3], [4].)

Recall that a stable mapping is defined as follows [1]. Let X and Y be smooth manifolds, $f$ and $f'$ be two elements from $\mathrm{C}^\infty(\mathrm{X},\mathrm{Y})$. Mappings $f$ and $f'$ are called equivalent, if there exist diffeomorphisms $g : \mathrm{X} \to \mathrm{X}$, and $h : \mathrm{Y} \to \mathrm{Y}$, such that

$$h \circ f = f' \circ g.$$

**Definition 1.** A mapping $f \in \mathrm{C}^\infty(\mathrm{X},\mathrm{Y})$ is called stable if there exists a neighborhood $W_f$ of $f$ in $\mathrm{C}^\infty(\mathrm{X},\mathrm{Y})$, such that any map $f' \in W_f$ is equivalent to $f$.

In other words, $f$ is stable if any map $f' \in W_f$ sufficiently close to $f$ may be transformed in to $f$ by suitable coordinate transformation in domain and in range. This imposes strong restrictions on the topology of mapping and its singular points [1]. We use this circumstance for obtaining some general results about the structure of the fibres of such mappings and suggest a fairly complete description of the possible topological structures of one- and two-dimensional fibres. We use general methods of singularity theory and our version of the signature method described above.

The precise statement of the problem under consideration is as follows. Let $Q : \mathbb{R}^s \to \mathbb{R}^t$ be a *stable proper quadratic mapping* (SPQM) with generic fiber of positive dimension $k = s - t$; we call such a mapping an SPQM of type $(s, t)$. The problem is to determine all possible topological structures of the fibres of the mapping $Q$ for $k = 1, 2$. The case where k = 0 is trivial since it is easy to show that each fiber is finite and contains at most $2^s$ points. The main results of this section refer to SPQMs of types $(4,3)$ and $(5,3)$, for which we were able to obtain a fairly complete description of the topology of fibres.For these dimensions, important topological information about fibres is given by the number of connected components and the Euler characteristic.

We proceed to describe the topology of low-dimensional fibers. The following two independent problems naturally arise: estimation of the number of components in a fiber and description of all possible topological types of the components. As to the former problem, it can be derived from general results from real algebraic geometry that the number of components in any fiber of a proper quadratic mapping of type $(s,t)$ as at most $2 \times 3^{k-1}$. For a particular mapping, this bound can be sharpened by constructing the bifurcation diagram of the mapping and calculating the number of components in the fiber over each component of the complement to the bifurcation diagram by known algorithmic methods.

For a stable quadratic mapping, the description of the topological structure of its fibers can be significantly refined by using the diverse data about the topology of stable mappings accumulated in the singularity theory.

In particular, as is known, the number of singular points in a fiber of a stable mapping of type $(s,t)$ does not exceed $t$ [1]. Combining this fact with the information about the local structure of singularities for stable mappings, we obtain strong constraints on the topological structure of the fibers. In this way, we obtained the following result for SPQMs of type (4,3).

**Theorem 1.** *The Euler characteristics of the fibers of a stable proper quadratic mappings $Q:\mathbb{R}^4 \to \mathbb{R}^3$ fill the integer segment $[-3,3] \bigcap \mathbb{Z}$. The complete list of possible homotopy types of the connected components of fibers comprises the point, the circle, the wedge of two circles, and the wedge of three circles.*

As is known, for some dimensions, complete lists of the topological types of fibers for stable mappings have been obtained. Possessing such a list, we can try to determine which possibility realizes for a given quadratic mappings. We can obtain complete lists of the topological types of fibers for stable quadratic mappings of type (4,3) and classify them up to isomorphisms of fibers as real algebraic manifolds. The lists obtained are fairly long, and we do not present them here. Moreover, an upper bound for the sum of multiplicities of the singular points in the fibers of an SPQM of given type can be found. Possibly, there exist some other constraints on the coexistence of singular points in one fiber, but we have not yet completed the study of this question. Similar results on the structure of fibers can be obtained for SPQMs of type (3,2), but for these dimensions, of most interest is a description of the possible types of embedding (knotting) of fibers in the ambient space, which has not been obtained so far.

The above considerations and results turn out to be useful for determining the stability properties of a given quadratic mapping. For example, consider the proper quadratic mapping $Q_4 : \mathbb{R}^4 \to \mathbb{R}^3$ determining the configuration space of a plane quadrangle. Let us show that it is unstable. The components of the mapping $Q_4$ in coordinates $(x_1,x_2,x_3,x_4)$ have the form

$$(x_1-1)^2+x_2^2, \;\; (x_3-x_1)^2+(x_4-x_2)^2, \;\; x_3^2+x_4^2.$$

For example, the configuration space of a quadrangle with sides $(1,l_2,l_3,l_4)$ is diffeomorphic to the fiber of $Q_4^{-l}(l)$ over the point $l=(l_2,l_3,l_4)$.

Using elementary geometric considerations, it is easy to show that the configuration space of the square (i.e., the fiber $Q_4^{-1}(1,1,1)$) is homeomorphic to the union of three circles pairwise tangent to each other at one point. This space is missing from the list of homeomorphism classes of the fibres of stable quadratic mappings; therefore, the mapping $Q_4$ is unstable. An analysis of this example suggests that the quadratic mappings determining the configuration spaces of equilateral plane $2k$-gons are unstable as well. Similar results are valid for the fibers of SPQMs of type (5,4), and they suggest that the Euler characteristics of the fibers of SPQMs of type $(s, s-1)$ fill the integer segment $[-s+1, s-1] \bigcap \mathbb{Z}$, but we were not able to prove this conjecture for all $s$.

For the two-dimensional fibers of SPQMs of type $(s, s-2)$, where $s$ is sufficiently small, similar assertions are valid. As an example, we state the result for SPQMs of type (5,3) (a complete description of the topology of fibers of SPQMs of type (4, 2) can be obtained by using the results of [5]).

**Theorem 2.** *The Euler characteristics of the fibres of any stable proper quadratic mapping $Q : \mathbb{R}^5 \to \mathbb{R}^3$ belong to the segment $[-6, 4]$. The connected components of the fibers are homeomorphic to compact Riemann surfaces of genus at most 4 on which some cycles not homologically equivalent to zero are contracted to points, and the number of singular points in each component does not exceed 3.*

The same methods apply to studying the fibers of SPQMs of type (6, 4). The latter theorem can be used according to the scheme described above to determine the instability of quadratic mappings related to the configuration spaces of some graphs.

Now we will describe the topology of typical fibres of stable quadratic maps when $n = 2, 3, 4, 5$ and $p = 3$. Thus we will deal with the stable intersections of three quadrics. Notice that the topology of intersection of two quadrics was in detail studied in [1]. The case $n = 2$ is very simple as we have three curves of second order and it is obvious that they can intersect at most in four points. So, for $Q : \mathbb{R}^2 \to \mathbb{R}^3$ the possible values of the number of components are $0, 1, 2, 3, 4$.

We proceed by estimating the number of connected components of a fibre for $n = 3$.

**Proposition 1.** *For $n = 3$, the fibres are finite and the number of points in a fibre fills in the integer segment $[0, 8] \bigcap \mathbb{Z}$.*

This follows by applying Bezout theorem and constructing explicit examples. More precisely, to each number from the integer segment $[0, 8] \bigcap \mathbb{Z}$ corresponds an obvious realization, therefore Euler characteristic takes all integer values from $[0, 8]$.

For $n = 4$, from dimensional reasons it follows that the generic fibre is a one-dimensional manifold. From properness follows that it is a finite union of simple closed curves.

**Proposition 2.** *The component number of any fibre does not exceed 54.*

This follows by explicating the general upper estimate $d(2d-1)^{n-1}$ for the components number of a real algebraic set of algebraic degree $d$ .

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Author's address:

T. Aliashvili
Department of Mathematics
Georgian Technical University
54, M. Kostava St., Tbilisi 0160
Georgia
E-mail: aliashvili@yahoo.com