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STUDY OF STRESS-STRAIN STATE OF THE PIECEWISE HOMOGENEOUS ELASTIC BODY

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Abstract. In the piecewise homogeneous elastic infinite body, with a circular hole and the radial cracks emanating from a surface this circular hole, is investigated the dependences of deformation on materials a body (circles with radius r and with the centre in tops of cracks consists of other material), on length the radius r, on the number and length of cracks. For some values of radius and length of cracks, numerical solutions are received by the boundary element method and corresponding graphs are constructed.

Keywords and phrases: Boundary element method, fictitious load, displacement discontinuity, crack, hole.

AMS subject classification: 65N38, 74B05, 74S15.

1. Introduction. In the literature on cracks emanating from the surfaces of circular holes the values of critical loads causing the development of cracks are calculated. In [l-3], problems of the limiting equilibrium are solved in closed form for a brittle plate weakened by a circular hole with one or two small linear cracks located at the ends of the hole.

The present paper and the my earlier works [4,5] deal with the question whether cracks can be helpful in strengthening structures. For example, when building underground structures, tunnels in particular, engineers intentionally make so-called technical cracks in the tunnel walls in order to decrease the stress concentration and fortify the cracks tops, using various techniques.

In [4,5] we investigated how the number of cracks and their lengths influence the stress distribution in the tunnel walls, i.e., how the tangential stress concentration on the circular [4] and elliptical [5] hole contour can be diminished by varying the number of cracks and their lengths.

In this paper we study of stress-strain state of the piecewise homogeneous elastic infinity body with a circular hole, when the internal circular cylindrical body surface contains four cracks and top of cracks are fortify to more rigid materials. Mathematical model of this practical problem represents boundary-contact problem, which is considered for an infinite body with circular hole, which internal the surface contains four radial cracks (Fig. 1), lengths of cracks are equal, thus, in some area at tips of cracks (a circle with radius r) and the others a body part, elastic characteristics (E, ν) strongly differ with each other. We deal with plane deformation. It is assumed that the body is free from internal stresses and that the along axis y tensile force is given at infinity.

Since here we deal with plane deformation, therefore the corresponding two-dimensional boundary-contact problem which is presented on Fig. 1 is considered.

2. Statement of a problem. From for symmetry we will set the task for a quarter of considered area (see Fig. 1, case b). Equilibrium equations will have the form:

$$\sigma_{ji,j}^{(g)} = 0, \quad i, j, g = 1, 2, \tag{1}$$

Hooke's law write a following kind

$$\sigma_{ij}^{(g)} = \sigma_{ji}^{(g)} = 2\mu^{(g)} \left(e_{ij}^{(g)} + \frac{\nu^{(g)}}{1 - 2\nu^{(g)}} e_{kk}^{(g)} \delta_{ij}^{(g)} \right), \tag{2}$$

where $e_{ij}^{(g)} = \frac{1}{2}\left(u_{i,j}^{(g)} + u_{j,i}^{(g)}\right)$ are deformations; $\mu^{(g)} = \frac{E^{(g)}}{2\left(1-\nu^{(g)}\right)}$; $\nu^{(g)}, E^{(g)}$ are known constants; δ is Kronecker symbol; $u_1^{(g)} = u^{(g)}$, $u_2^{(g)} = v^{(g)}$ components of a displacement vector; $\sigma_{ji}^{(g)}$ are components of stress tensor; g is number of the domain ($g = 1$ for Ω, $g = 2$ for Ω_1, see Fig. 1,b)

The boundary conditions have the form:

On Γ_q and on the crack AB

$$\sigma_n^{(g)} = 0, \quad \sigma_s^{(g)} = 0, \tag{3}$$

$$y \to \infty: \qquad \sigma_n^{(1)} = P, \qquad \sigma_s^{(1)} = 0. \tag{4}$$

On boundaries $x = 0$ and $y = 0$ symmetry conditions will be satisfied [6] (when $x = 0$ then $u^{(g)} = 0$, $\sigma_s^{(g)} = 0$; when $y = 0$ then $v^{(g)} = 0$, $\sigma_s^{(g)} = 0$).

Contact conditions have the form:

On a circle Γ_2 rigid contact conditions is carried out

$$\sigma_n^{(1)} = \sigma_n^{(2)}, \quad \sigma_s^{(1)} = \sigma_s^{(2)}, \quad u^{(1)} = u^{(2)}, \quad v^{(1)} = v^{(2)}. \tag{5}$$

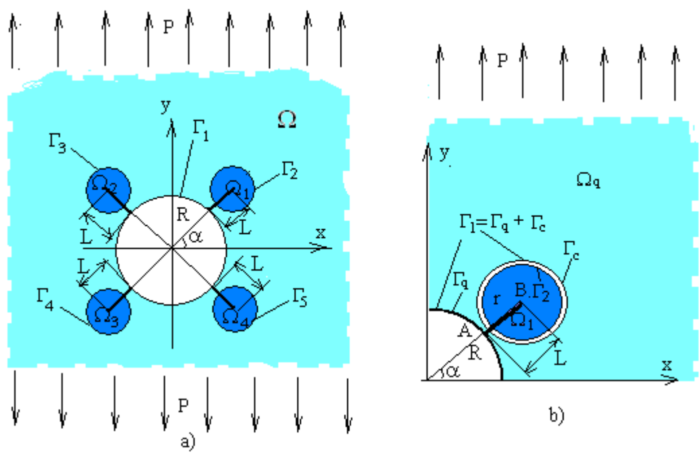

Fig. 1. Geometry and loading of a) infinite area with a circular hole and four cracks emanating from him and b) of a quarter of this area

3. Decision of a boundary-contact problem. (1)-(5) boundary-contact problem numerical solutions are received by the boundary element method, in particular, by the combined method of fictitious load method [7] and displacement discontinuity method [8]. Boundary element equations can be written as [7]:

$$\left.\begin{aligned} d_s^i &= \sum_{j=1}^{N} C_{ss}^{ij} P_s^j + \sum_{j=1}^{N} C_{sn}^{ij} P_n^j \\ d_n^i &= \sum_{j=1}^{N} C_{ns}^{ij} P_s^j + \sum_{j=1}^{N} C_{nn}^{ij} P_n^j \end{aligned}\right\}, \quad i = 1, \dots, N. \tag{6}$$

Quantity of elements which belongs to boundary and contact contours is $N = N_1 + N_2$, where quantities of elements which belongs to Γ_1, and Γ_2 contours is N_1 and N_2 respectively.

Let's notice, that typical i element should situated or on a free part of one of boundary contours, or on a surface of contact of two subareas and or on cracks. In the first case i-th equation in (6) turns out proceeding from boundary conditions (3), (4), and they are in the second case by use of conditions of a continuity on a contact surface. Thus, for $d_s^i, d_n^i, C_{nn}^{ij}, \dots$, from (6) the following expression are obtained.

let us assume, that the i-th element is lying on a free part of a contour Γ_1. If accept further, that on this element are given stresses, then we receive

$$d_s^i = \sigma_s^{i(1)}, \qquad d_n^i = \sigma_n^{i(1)},$$
$$C_{ss}^{ij} = \begin{cases} A_{ss}^{ij(1)}, & j \le N_1, \\ 0, & N_1 + 1 \le j \le N. \end{cases}$$

Analogously expressions are obtain for C_{sn}^{ij}, C_{ns}^{ij} and C_{nn}^{ij} coefficients and in the event that the i-th element is lying on a free part of a contour Γ_1 and if on it displacements are given.

Let's consider a case when the i-th element lays on a surface of contact of two subareas. The element on a contact surface is considered virtually as two conterminous boundary elements, one for a contour Γ_1 and another for a contour Γ_2. Thus, if the i-th element belongs to one side of a surface of contact, there is pair to it an i^* element, say, on other side. On each element laying on a surface of contact, four conditions should be satisfied also. Two of them can be used for composed of the equations concerning an element on one side of a surface of contact, and two others can be used analogously for an element on other side. Thus, for elements laying on a surface of contact of $d_s^i, d_n^i, C_{nn}^{ij}, \dots$ (5) are expressed by following formulas.

If the i-th element lays on a contour Γ_1, and the i^* element lays on a contour Γ_2 and on this element are given stresses, then

$$d_s^i = \sigma_s^{i(1)} - \sigma_s^{i^*(2)}, \qquad d_n^i = \sigma_n^{i(1)} - \sigma_n^{i^*(2)}, \qquad C_{ss}^{ij} = \begin{cases} A_{ss}^{ij(1)}, & j \le N_1 \\ -A_{ss}^{i^*j(2)}, & N_1 + 1 \le j \le N_1 + N_2, \end{cases}$$

etc. for coefficients C_{sn}^{ij}, C_{ns}^{ij} and C_{nn}^{ij}.

If the i-th element lays on a contour Γ_2, and the i^* element lays on a contour Γ_1 and on this elements are given displacements, then

$$d_s^i = v^{i(2)} + v^{i^*(1)}, \qquad d_n^i = u^{i(2)} + u^{i^*(1)}, \qquad C_{ss}^{ij} = \begin{cases} B_{ss}^{i^*j(1)}, & j \le N_1, \\ B_{ss}^{ij(2)}, & N_1 + 1 \le j \le N_1 + N_2. \end{cases}$$

Analogously expressions are obtain for the rest coefficients.

When the element lays on a crack then is obtain of the same system towards the displacement discontinuity D_s^j, D_n^j [6].

For the boundary conditions to be satisfied on Γ_1 or Γ_2, we use formulas obtained by the method of fictitious loads, while for cracks we use formulas obtained by the displacement discontinuity method. Thus we come to the system linear equations with unknowns P_s^j, P_n^j and D_s^j, D_n^j.

After solving system we can calculate displacements and stresses at any point of the body except the points lying inside the circle with center in the middle of a boundary element and of radius equal to the length of this element, certainly not counting the midpoint of the element.

4. Numerical solutions and their discussions. Numerical solutions above-stated problems for following data are received: $E^{(1)} = 2 \cdot 10^3\,(kg/m^2)$, $\nu^{(1)} = 0.38$, $E^{(2)} = 2 \cdot 10^6\,(kg/m^2)$, $\nu^{(2)} = 0.3$, $\alpha = \pi/4$, $P = 10(kg)$, $N_1 = 50$, $\mathrm{N}_2 = 50$, $R = 4\ (m)$, $L = 1,\ 2,\ 3,\ 4\,(m)$ and $r = 0,4;\ 0,5;\ 0,8;\ 1;\ 1,2;\ \ 1,8;\ 2;\ 2,8;\ 3;\ 3,6\,(m)$. In particular, numerical values of strains and displacements are received in considered body and corresponding graphs are constructed (see Tab. 1 and Fig.2). After discussion of the received results and corresponding graphs, some are chosen from them which are presented below. As the harden crack top is the purpose of a problem, therefore the variation of radius r of a circle around top (which of other material) and a crack length$L$, in a body (especially around top) decreases concentration of tangential stresses. On Fig. 2 are presented depiction which shows distributions of tangential stresses in a body for following data $L = 2\,(m)$ and $r = 0,5\,(m)$ or $1,2\,(m)$; or $1,8\ .(m)$.

Table 1. The maximal numerical values of the tangential stress and the displacements for some values of L and r.

	$L=1$		$L=2$			$L=3$		$L=4$	
	$r=0,4$	$r=0,8$	$r=0,5$	$r=1,2$	$r=1,8$	$r=1,8$	$r=1$	$r=1$	$r=1$
σ_t	315,0837	420,3393	262,1601	568,7799	811,5163	316,0829	583,4944	376,9433	548,9905
u	0,350211	0,661409	0,427695	1,073365	1,215883	0,385061	0,854240	1,016821	1,859833
v	-0,1396	-0,2996	-0,1948	-0,5033	-0,9099	-0,1543	-0,3486	-0,4598	-0,9162

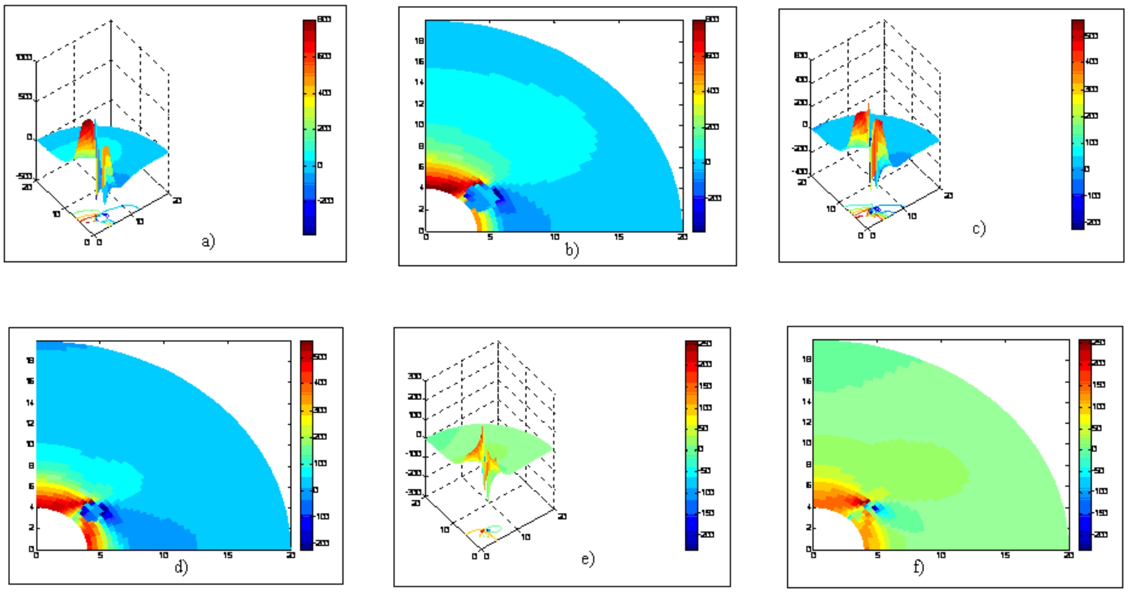

Fig. 2. The change tangential stresses in a quarter of an infinite body which contains circular hole with radius R, and crack, when length of a crack $L = 2$, a) $r = 1,8$; b) pseudo color plot, $r = 1,8$; c) $r = 1,2$; d) pseudo color plot, $r = 1,2$; e) $r = 0,5$; f) pseudo color plot, $r = 0,5$.

R E F E R E N C E S

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