Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 24, 2010

THE SOLUTION OF THE STRESS BOUNDARY VALUE PROBLEM OF ELASTOSTATICS FOR DOUBLE POROUS PLANE WITH A CIRCULAR HOLE

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Abstract. In the present work we solve explicitly, by means of absolutely and uniformly convergent series, the second boundary value problem of porous elastostatics for the plane with a circular hole. For the particular boundary value problem the numerical results is given.

Keywords and phrases: Porous media, double porosity, stress boundary value problem, numerical solution.

AMS subject classification: 74F10, 74G10, 74G15.

We consider the plane D with a circular hole. Let R be the radius of the boundary S. The system of equations of porous elastostatics is of the form [1]:

$$\begin{array}{l} \mu\Delta(u(x)) + (\lambda+\mu) graddiv(u(x)) = grad[\beta_1 p_1(x) + \beta_2 p_2(x)], \\ (m_1\Delta - k)p_1(x) + kp_2(x) = 0, \\ kp_1(x) + (m_2\Delta - k)p_2(x) = 0, x \in D, \end{array} \tag{1}$$

where $\lambda, \mu, m_1, m_2, \beta_1, \beta_2$ are the known elastic and physical constants [1,2]; $u(x) = (u_1(x)), u_2(x))$ is the displacement of the point x; p_1 is the fluid pressure within the primary pores and p_2 is the fluid pressure within the secondary pores; Δ is the Laplace operator.

Problem. Find a regular solution $U(u_1, u_2, p_1, p_2)$ of system (1) satisfying the boundary conditions

$$P(\partial_z, n)U(z) = f(z), \frac{\partial p_1(z)}{\partial n} = f_3(z), \frac{\partial p_2(z)}{\partial n} = f_4(z), z \in S, \tag{2}$$

where

$$P(\partial_x, n)U(x) = T(\partial_x, n)u(x) - n(x)[\beta_1 p_1(x) + \beta_2 p_2(x)]$$

is the stress vector of the theory poroelasticity; $T(\partial_x, n)u(x) = \mu\partial_n u(x) + \lambda n(x) div(u(x))$ $+\mu \sum\limits_{i=1}^{\infty} n_i(x) gradu_i(x)$ is the stress vector of the theory of elasticity; $f(z) = (f_1(z), f_2(z))$, $f_3(z), f_4(z)$ are the given functions on the circumference S, $n = n(n_1, n_2)$. Vector $U(x)$ satisfies the following conditions at infinite:

$$u(x) = O(1), \quad r^2\partial_{x_k} u(x) = O(1), \quad r^2 p_i(x) = O(1), \quad k = 1, 2,$$

where $r^2 = x_1^2 + x_2^2$. On the basic of the system (1), we can write

$$p_1 = a_1\varphi_1(x) + p_1 + a_2\varphi_2(x), p_2 = a_3\varphi_1(x) + p_1 + a_4\varphi_2(x) \tag{3}$$

where $\Delta\varphi_1 = 0, (\Delta + \lambda_0^2)\varphi_2 = 0, \lambda_0^2 = -\dfrac{k(m_1+m_2)}{m_1 m_2}, a_1 = a_3 = \dfrac{2}{m_1+m_2},$
$a_2 = -\dfrac{m_1 - m_2}{m_1(m_1+m_2)}, a_4 = \dfrac{m_1 - m_2}{m_2(m_1+m_2)}, k, m_1, m_2 > 0.$

Using (3), the conditions (2) allow us to find the values of the functions φ_1 and φ_2 on S;

$$\partial_R\varphi_1(z) = \Omega_1(z), \partial_R\varphi_2(z) = \Omega_2(z), z \in S,$$

where $\Omega_1(z) = \frac{d_1}{d}, \Omega_2(z) = \frac{d_2}{d}, d = a_1a_4 - a_2^2, d_1 = a_4f_3 - a_2f_4, d_2 = a_1f_4 - a_2f_3, \partial_n = [\partial_r]_{r=R}, r^2 = x_1^2 + x_2^2$. The harmonic function $\varphi_1(x)$ is defined by the series

$$\varphi_1(x) = c - \sum_{m=1}^{\infty} \frac{R}{m}\left(\frac{R}{r}\right)^m (A_m\cos(m\psi) + B_m\sin(m\psi), \tag{4}$$

where $x = (r, \psi)$; A_m and B_m are the coefficients of the Fourier series for the known function $\Omega_1(z)$.

The metaharmonic function $\varphi_2(x)$ is defined by the series [3]:

$$\varphi_2(x) = \sum_{m=1}^{\infty} \frac{K_m(\lambda_0 r)}{\lambda_0 K'_m(\lambda_0 R)}(C_m\cos(m\psi) + D_m\sin(m\psi)), \tag{5}$$

where $K_m(\lambda_0 r)$ is the MacDonald's function with an imaginary argument; C_m and D_m are the Fourier coefficients for the known function $\Omega_2(z)$;$K'_m(\varsigma) = \partial_\varsigma K_m(\varsigma), \partial_r K_m(\lambda_0 r) = \lambda_0 K'_m(\lambda_0), K'_m(\lambda_0) \neq 0$.

Thus by means of (3), the functions φ_1 and φ_2 are defined explicitly.

The solution of the first equation of the system (1) with the condition (2) is given by the sum

$$u(x) = v_0(x) + v(x), \tag{6}$$

where v_0 is the particular solution of equation $(1)_1$,

$$v_0(x) = \frac{1}{\lambda + 2\mu} grad(-\frac{a}{\lambda_0^2}\varphi_2 + b\varphi_0), \tag{7}$$

φ_0 is the biharmonic function:$\Delta\varphi_0 = \varphi_1$;

$$\varphi_0(x) = \frac{R^3}{4}\sum_{m=2}^{\infty}(\frac{1}{m(1-m)}\left(\frac{R}{r}\right)^{m-2} (A_m\cos(m\psi) + B_m\sin(m\psi)), \tag{8}$$

$a = (\beta_1 + \beta_2)a_1, b = \beta_1 a_2 + \beta_2 a_4$; A_m and B_m are given by (4).

v is the solution of the homogeneous equation which can be found by means of the formula

$$v(x) = grad[\Phi_1(x) + \Phi_2(x)] + rot\Phi_3(x), \tag{9}$$

where $\triangle\Phi_1(x) = 0, \triangle\triangle\Phi_2(x) = 0, \triangle\triangle\Phi_3(x) = 0, rot = (-\partial_{x_2}, \partial_{x_1})$,
$\Phi_1(x) = \sum\limits_{m=0}^{\infty}\left(\frac{R}{r}\right)^m (X_{m1}\cdot\nu_m(\psi)), \Phi_2(x) = \sum\limits_{m=0}^{\infty}\left(\frac{R}{r}\right)^{m-2} R^2(X_{m2}\cdot\nu_m(\psi)),$

$$\Phi_3(x) = \frac{R^2(\lambda+2\mu)}{\mu}\sum_{m=0}^{\infty}\left(\frac{R}{r}\right)^{m-2}(X_{m2}\cdot s_m(\psi)), \nu_m(\psi) = (\cos(m\psi), \sin(m\psi),$$

$s_m(\psi) = (-\sin(m\psi), \cos(m\psi));$

$$X_{01} = \frac{\alpha_0}{4(\lambda+2\mu)}, X_{02} = \frac{\beta_0}{4(\lambda+2\mu)};$$

X_{m1} and X_{m2} is the solution of the following system:

$$m[\lambda+2\mu(m+1)]X_{m1}$$

$$+\{(\lambda+2\mu)(1-m)(2-m+\frac{\lambda+2\mu}{\mu}m)-\lambda\cdot m\cdot R^2[m+\frac{\lambda+2\mu}{\mu}(2-m)]\}X_{m2} = \alpha_m R^2,$$

$$-m(1+2\mu)X_{m1} + R^2[m(3-2m)+\frac{\lambda+2\mu}{\mu}(m^2-3m+2)]X_{m2} = \beta_m\frac{R^2}{\mu},$$

$$m = 1, 2, ...;$$

α_m and β_m are the Fourier coefficients of, respectively, the normal and tangential components of the function $\Psi(z) = f(z) + n(z)[a\varphi_2(z) + b\varphi_1(z)] - T(\partial_z, n)v_0(z)$.

For the numerical solution there is the program. $p_1(x)$ and $p_2(x)$ are calculated from (3), (4) and (6); $u_1(x)$ and $u_2(x)$ are calculated from (6), where $v_0(x)$ calculated from (7), (5) and (8), while $v(x)$ from (9).

Let us consider a particular case with the following conditions:

$$R = 2; r = 3.2; \psi = 60^\circ; \lambda = 7.28\cdot 10^6; \mu = 3.5\cdot 10^6; m_1 = 0.88; m_2 = 0.22; k = 1;$$

$$\beta_1 = 0.3;\quad \beta_2 = 0.4;\quad f_1(\theta) = 5R(2\cos\theta+3);\quad f_2(\theta) = 10R(5\sin\theta - 7);$$

$$f_3(\theta) = \frac{R}{3}(\cos\theta - 0.1)\cdot 10^{-1};\quad f_4(\theta) = \frac{3R}{4}(\sin\theta+0.1), 0\le\theta\le 2\pi.$$

We obtain that:

$$u_1 = 4.246\cdot 10^{-5};\quad u_2 = 1.529\cdot 10^{-5};\quad p_1 = -0.024,\quad p_2 = -0.156.$$

Acknowledgment. The designated project has been fulfilled by financial support of Georgian National Science Foundation (Grant GNSF/ST 08/3-088). Any idea in this publication is possessed by the author and may not represent the opinion of Georgian National Science Foundation itself.

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Received 10.06.2010; revised 19.09.2010; accepted 28.10.2010.

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