Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 24, 2010

ROBUST MEAN-VARIANCE HEDGING IN THE TWO PERIOD MODEL

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Abstract. We provide an explicit solution of robust mean-variance hedging problem in the two period model for some type of contingent claims.

Keywords and phrases: Minimax problem, mean-variance hedging, maximum principle, robust optimization.

AMS subject classification: 60H30, 90C47.

The study of mean variance hedging problem was initiated by H. Föllmer and D. Sondermann [1] and the solution of this problem for multiperiod model was given by H. Föllmer and M. Schweizer [2]. In this paper we investigate two period mean variance hedging problem of contingent claims in incomplete markets, when parameters of asset prices are not known with certainty. Usually such parameters may be appreciation rate (or drift) and volatility coefficients. In such models it is desirable to choose an optimal portfolio for the worst case of the coefficients. Such type problem one calls the robust hedging problem.

The purpose of the present paper is to investigate the robust mean-variance hedging problem in the two period model, when drift and volatility of the asset are not known exactly. We consider the minimax problem and construct the optimal strategy for some type of contingent claims. The main approach we develop is the randomization of the parameters and change the minimax problem by maximin one. This approach successfully works in the two period model and preliminary results show that it will be productive in multi-period and continuous time models.

Let $(S_t, \eta_t),\ t = 0, 1$ be the price of assets. We suppose that

$$S_1 = S_0 + \mu + \sigma w,\ \eta_1 = \beta + \delta \bar{w},$$

where $w, \bar{w}$ is random pair with $Ew = E\bar{w} = 0,\ Dw = D\bar{w} = 1,\ Cov(w, \bar{w}) \neq 0$ and $\mu, \sigma, \beta, \delta$ are constants. We suppose also that the appreciation rate μ and volatility σ of the asset price S_t are misspecified but stay in rectangle of uncertainty, i.e.

$$(\mu, \sigma) \in D = [\mu_-, \mu_+] \times [\sigma_-, \sigma_+].$$

Let β, δ be known exactly. We denote by π the number of stocks S bought at time $t = 0$ and by $x_0 = \pi S_0$ the initial capital. The wealth at time $t = 1$ is

$$X_1 = x_0 + \pi(S_1 - S_0) = x_0 + \pi\mu + \pi\sigma w. \tag{1}$$

The contingent claim $H(\eta)$ we assume depends on the asset η, which cannot be traded directly. The robust mean-variance hedging problem is

$$\min_{\pi \in R} \max_{(\mu,\sigma)\in D} E|H - x_0 - \pi\mu - \pi\sigma w|^2. \tag{2}$$

Let

$$H - x_0 = h_0 + h_1 w + H^{\perp}$$

be the decomposition of $H - x_0$ with $h_0 = E(H - x_0),\ h_1 = EwH,\ EH^{\perp} = 0,\ EwH^{\perp} = 0$. Then the problem can be rewritten as

$$\min_{\pi \in R} \max_{(\mu,\sigma)\in D} F(\pi, \mu, \sigma),$$

where

$$F(\pi, \mu, \sigma) = (h_0 - \pi\mu)^2 + (h_1 - \pi\sigma)^2.$$

The function $F(\pi, \cdot)$ can be continued on the space of probability measures on D as

$$F(\pi, \nu) = \int_D ((h_0 - \pi\mu)^2 + (h_1 - \pi\sigma)^2)\nu(d\mu d\sigma), \quad \text{for a measure } \nu \text{ on } D.$$

Hence we get

$$\begin{aligned} F(\pi, \nu) = \int_D (\mu^2 + \sigma^2)\nu(d\mu d\sigma) \left(\pi - \frac{\int_D (h_0\mu + h_1\sigma)\nu(d\mu d\sigma)}{\int_D (\mu^2 + \sigma^2)\nu(d\mu d\sigma)}\right)^2 \\ + h_0^2 + h_1^2 - \frac{(\int_D (h_0\mu + h_1\sigma)\nu(d\mu d\sigma))^2}{\int_D (\mu^2 + \sigma^2)\nu(d\mu d\sigma)} \end{aligned}$$

and

$$\min_{\pi \in R} F(\pi, \nu) = h_0^2 + h_1^2 - \frac{(\int_D (h_0\mu + h_1\sigma)\nu(d\mu d\sigma))^2}{\int_D (\mu^2 + \sigma^2)\nu(d\mu d\sigma)},$$

$$\pi^* = \frac{\int_D (h_0\mu + h_1\sigma)\nu(d\mu d\sigma)}{\int_D (\mu^2 + \sigma^2)\nu(d\mu d\sigma)}.$$

Since F is strictly convex in π by the Theorem Neumann at al. (see Theorem lX.4.1 of [3]) there exists a saddle point (π^*, ν^*), i.e.

$$F(\pi^*, \nu) \le F(\pi^*, \nu^*) \le F(\pi, \nu^*).$$

Since $\max_\nu F(\pi, \nu) = \max_{\mu,\sigma} F(\pi, \mu, \sigma)$ then we obtain

$$\min_{\pi} \max_{(\mu,\sigma)} F(\pi, \mu, \sigma) = \min_{\pi} \max_{\nu} F(\pi, \nu) = \max_{\nu} \min_{\pi} F(\pi, \nu). \tag{3}$$

Each pair of random variables (μ, σ) with the distribution ν may be realized on the probability space $([0.1], \mathcal{B}, P(d\omega) = d\omega)$ where $\mathcal{B}$ is the Borel $\sigma-$algebra on $[0, 1]$ and $d\omega$ the Lebesgue measure (see Proposition 26.6 of [4]). Hence the minimization problem

$$\min_{\nu} \frac{(\int_D (h_0\mu + h_1\sigma)\nu(d\mu d\sigma))^2}{\int_D (\mu^2 + \sigma^2)\nu(d\mu d\sigma)}$$

can be written as

$$\min_{(\mu(\omega),\sigma(\omega))\in D} \frac{(\int_0^1 (h_0\mu(\omega) + h_1\sigma(\omega))d\omega)^2}{\int_0^1 (\mu^2(\omega) + \sigma^2(\omega))d\omega}.$$

To solve this problem we consider the deterministic control problem

$$\max_{(\mu(\omega),\sigma(\omega))\in D}\int_0^1(\mu^2(\omega)+\sigma^2(\omega))d\omega, \tag{4}$$

$$\frac{dx(\omega)}{d\omega}=\mu(\omega)\quad \frac{dy(\omega)}{d\omega}=\sigma(\omega), \tag{5}$$

$$x(0)=0,y(0)=0,\ \ x(1)=x,y(1)=y. \tag{6}$$

Lemma. *The solution of the problem (4)-(6) is of the form*

$$\mu^*(\omega)=\mu_-\chi_A(\omega)+\mu_+\chi_{A^c}(\omega),\ \ \sigma^*(\omega)=\sigma_-\chi_B(\omega)+\sigma_+\chi_{B^c}(\omega), \tag{7}$$

with

$$P(A)=\frac{x-\mu_-}{\mu_+-\mu_-},\ \ P(B)=\frac{y-\sigma_-}{\sigma_+-\sigma_-} \tag{8}$$

and the maximal value is $2x\mu_M+2y\sigma_M-\mu_-\mu_+-\sigma_-\sigma_+$*, where* $\mu_M=\frac{\mu_++\mu_-}{2}$, $\sigma_M=\frac{\sigma_++\sigma_-}{2}$.

Proof. By the maximum principle (see [3]) we have

$$\mu^*=\arg\max_{\mu_-\le\mu\le\mu_+}(\mu^2+p\mu),\ \sigma^*=\arg\max_{\sigma_-\le\sigma\le\sigma_+}(\sigma^2+q\sigma),$$

where p,q are some constants maintaining the conditions (6). Hence the solution of the problem (4)-(6) is of the form (7), (8). The relations

$$\int_0^1\mu^*(\omega)d\omega=x,\ \ \int_0^1\sigma^*(\omega)d\omega=y$$

uniquely determine the probabilities $P(A),P(B)$ by (8) and

$$\int_0^1(\mu^{*2}(\omega)+\sigma^{*2}(\omega))d\omega=2x\mu_M+2y\sigma_M-\mu_-\mu_+-\sigma_-\sigma_+.$$

Corollary.

$$\min_{(\mu(\omega),\sigma(\omega))\in D}\frac{(\int_0^1(h_0\mu(\omega)+h_1\sigma(\omega))d\omega)^2}{\int_0^1(\mu^2(\omega)+\sigma^2(\omega))d\omega}=\min_{(x,y)\in D}\frac{(h_0x+h_1y)^2}{2\mu_Mx+2\sigma_My-\mu_-\mu_+-\sigma_-\sigma_+}.$$

Finally we obtain

Theorem. *The solution of minimax problem (3),* (π^*,μ^*,σ^*) *is given by the formulae*

$$\pi^*=\frac{h_0x^*+h_1y^*}{2\mu_Mx^*+2\sigma_My^*-\mu_-\mu_+-\sigma_-\sigma_+},$$

$$P(\mu^*=\mu_-)=\frac{\mu_+-x^*}{\mu_+-\mu_-},\ P(\sigma^*=\sigma_-)=\frac{\sigma_+-y^*}{\sigma_+-\sigma_-},$$

$$P(\mu^*=\mu_+)=\frac{x^*-\mu_-}{\mu_+-\mu_-},\ P(\sigma^*=\sigma_+)=\frac{y^*-\sigma_-}{\sigma_+-\sigma_-},$$

where

$$(x^*, y^*) = \arg \min_{(x,y)\in D} \frac{(h_0 x + h_1 y)^2}{2\mu_M x + 2\sigma_M y - \mu_- \mu_+ - \sigma_- \sigma_+}.$$

Moreover π^* is the solution of the problem (1), (2).

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Received 13.06.2010; revised 19.10.2010; accepted 18.11.2010.

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