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SOLUTION OF THE BASIC MIXED BOUNDARY VALUE PROBLEM OF STATICS IN THE LINEAR THEORY OF ELASTIC MIXTURE FOR AN HALF-PLANE

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Abstract. By the method N. Muskhelishvili an explicit solution to the basic mixed boundary value problem for homogeneous equation of statics of the linear theory of elastic mixture for an lower half-plane is obtained.

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$\mathbf{1^0}$. A homogeneous equation of statics of the theory of elastic mixtures in a complex form is of the type [1]

$$\frac{\partial^2 U}{\partial z \partial \overline{z}} + K\frac{\partial^2 \overline{U}}{\partial \overline{z}^2} = 0, \quad U = (u_1 + iu_2, u_3 + iu_4)^T,$$

where $z = x_1 + ix_2$, $u' = (u_1, u_2)^T$ and $u'' = (u_3, u_4)^T$ are partial displasements,

$$K = -\frac{1}{2}em^{-1}, \quad e = \begin{bmatrix} e_4, & e_5 \\ e_5, & e_6 \end{bmatrix}, \quad m^{-1} = \begin{bmatrix} m_1, & m_2 \\ m_2, & m_3 \end{bmatrix}^{-1}, \quad \Delta_0 = \det m > 0;$$

m_k, e_{3+k}, $k = 1, 2, 3$, are expressed in terms of elastic constants [1].

In [2] M. Basheleishvili obtained the representations:

$$2\mu U = 2\mu(u_1 + iu_2, u_3 + iu_4)^T = A\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu\overline{\psi(z)},$$

$$TU = \big((Tu)_2 - i(Tu)_1 (Tu)_4 - i(Tu)_3\big) = \begin{pmatrix} r'_{12}n_1 + r'_{22}n_2 - i(r'_{11}n_1 + r'_{21}n_2) \\ r''_{12}n_1 + r''_{22}n_2 - i(r''_{11}n_1 + r''_{21}n_2) \end{pmatrix}$$

$$= \Big(n_1\frac{\partial}{\partial x_2} - n_2\frac{\partial}{\partial x_1}\Big)[(A - 2E)\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu\overline{\psi(z)}],$$

where $\varphi = (\varphi_1, \varphi_2)^T$ and $\psi = (\psi_1, \psi_2)^T$ are arbitrary analytic vector-functions; $(TU)_p$, $p = \overline{1, 4}$, are the components of stresses, $n = (n_1, n_2)$ is unit vector;

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = 2\mu m, \quad B = \mu e, \quad \mu = \begin{bmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{bmatrix}, \quad E = \begin{bmatrix} 1, & 0 \\ 0, & 1 \end{bmatrix},$$

μ_1, μ_2, μ_3 are elastic constants [1]; $\Delta_1 = \det\mu > 0$, $\Delta_2 = \det(A - 2E) > 0$.

Elementary calculations result in [3]

$$2\mu U' = 2\mu\frac{\partial U}{\partial x_1} = A\phi(z) + B\overline{\phi(z)} + Bz\overline{\phi'(z)} + 2\mu\overline{\Psi(z)}, \tag{1}$$

$$\overset{(1)}{\tau} + \overset{(2)}{\tau} = 2(2E - A - B)\operatorname{Re}\phi(z), \quad \overset{(1)}{\eta} - \overset{(2)}{\eta} = 2(A - B - 2E)\operatorname{Im}\Phi(z), \tag{2}$$

$$\overset{(1)}{\tau} - \overset{(2)}{\tau} - i(\overset{(1)}{\eta} + \overset{(2)}{\eta}) = 2[B\overline{z}\phi'(z) + 2\mu\Psi(z)], \tag{3}$$

$$\overset{(2)}{\tau} - i\overset{(1)}{\eta} = (2E - A)\phi(z) - B\overline{\phi(z)} - Bz\overline{\phi'(z)} - 2\mu\overline{\Psi(z)}, \tag{4}$$

where $\phi(z) = \varphi'(z)$, $\Psi(z) = \psi'(z)$, $\overset{(1)}{\tau} = (r'_{11}, r''_{11})^T$, $\overset{(2)}{\tau} = (r'_{22}, r''_{22})^T$, $\overset{(1)}{\eta} = (r'_{21}, r''_{21})^T$, $\overset{(2)}{\eta} = (r'_{12}, r''_{12})^T$.

2⁰. Let $D^+(D^-)$ be a $x_2 > 0$, $(x_2 < 0)$ region and L is $0x_1$ axis. Let exterior unit normal of the D^- is $n = (0,1)^T$. Suppose that the $L'_k = a_k b_k$, $k = \overline{1,n}$, their positive directions coincides with the $0x_1$ axis. Let $L = L' + L''$, where $L' = \bigcup\limits_{k=1}^{n} L'_k$ and L'' is a remaind part of the L.

Let an elastic plate of mixture occupied a domain D^-, and we consider the mixed boundary value problem. Define an elastic equilibrium of the plate if

$$2\mu U^-(t) = f(t), \quad t \in L', \quad (TU(t))^- = 0, \quad t \in L'',$$

where $f(t) = (f_1, f_2)^T$ is given vector, $f' \in H$, $f = a_0 + o(1)$, a_0 is a known constant vector.

The components of the stresses and rotation at infinity as well as the principal vector of external forces applied to the L'' will be assumed to zero. The value of the principal vector on the L'_k is equal to $P_k + iQ_k$, $k = \overline{1,n}$.

An analogous problem for simple connected domain have been considered in [4] and for an infinite plate with an alliptical hole in [5].

Using the Green formula [1] it is easy to prove that, the homogeneous mixed boundary value problem ($f = 0$) admits a trivial solution only.

On the basic of formulas (1)-(4) our problem is redused to finding two analutic vector-functions $\phi(z)$ and $\Psi(z)$ in D^- vanishing at infinity, by the boundary conditions:

$$A(\phi(t))^- + B\overline{(\phi(t))^-} + Bt\overline{(\phi'(t))^-} + 2\mu\overline{(\Psi(t))^-} = f'(t), \quad t \in L', \tag{5}$$

$$(A - 2E)(\phi(t))^- + B\overline{(\phi(t))^-} + Bt\overline{(\phi'(t))^-} + 2\mu\overline{(\Psi(t))^-} = 0, \quad t \in L''.$$

Now note that the equation

$$(A - 2E)\phi(z) = -B\overline{\phi(\overline{z})} - Bz\overline{\phi'(\overline{z})} - 2\mu\overline{\Psi(\overline{z})}, \qquad \text{for any} \quad z \in D^+, \tag{6}$$

define $\phi(z)$ as analytic vector-function toward z in the domain D^+, and to $\overline{z}$-in D^-.

By virtue of (6) we have

$$(A - 2E)(\phi(t))^+ = -B\overline{(\phi(t))^-} - Bt\overline{(\phi'(t))^-} - 2\mu\overline{(\Psi(t))^-}, \quad t \in L''. \tag{7}$$

Due to (6) and (7), we arrive to $(\phi(t))^+ = (\phi(t))^-$, $t \in L''$, where from it follows $\phi(z)$ is analytic in the entire $z = x_1 + ix_2$ plane cuted along to the L' furthermore in the neighborhood of points a_k and b_k $\phi(z)$ admit estimate of type

$$|\phi_j(t)| < \operatorname{const}|z - \alpha|^{-\delta}, \ \ 0 \le \delta < 1, \ \ (\alpha = a_k \quad \text{or} \quad \alpha = bk, \ \ k = \overline{1,n}), \ \ j = 1, 2.$$

In equality (6) we have

$$2\mu\Psi(z)=-(A-2E)\overline{\phi(\overline{z})}-B\phi(z)-Bz\phi'(z),\qquad \text{for any}\quad z\in D^{-}. \tag{8}$$

It follows from (8) that the vector-function $\Psi(z)$ is definite in the entire $z=x_1+ix_2$ plane by means of $\phi(z)$.

Taking now into account the last formula from (1), (2) and (3) we get

$$2\mu U=A\varphi(z)-(A-2E)\varphi(\overline{z})+B(z-\overline{z})\varphi'(z)+\text{const}, \tag{9}$$

$$2\mu U'=A\phi(z)-(A-2E)\phi(\overline{z})+B(z-\overline{z})\phi'(z), \tag{10}$$

$$\overset{(2)}{\tau}-i\overset{(1)}{\eta}=(A-2E)(\phi(\overline{z})-\phi(z))-B(z-\overline{z})\overline{\phi'(z)}. \tag{11}$$

In view of (10) the (5) boundary condition can be written as

$$(\phi(t))^{+}-(A-2E)^{-1}A(\phi(t))^{-}=-(A-2E)^{-1}f'(t)=R(t),\quad t\in L'. \tag{12}$$

A solution of the (12) problem can be represented it the form

$$\phi(z)=\frac{1}{y_2-y_1}\begin{bmatrix}y_2 & -1\\ -y_1 & 1\end{bmatrix}\left\{\frac{X(z)}{2\pi i}\int_{L'}\frac{[X^{+}(t)]^{-1}R(t)dt}{t-z}+X(z)P_{n-1}(z)\right\}, \tag{13}$$

where y_1 and y_2 are the roots of the equation $A_3y^2+(A_1-A_4)y-A_2=0$, $X(z)=\begin{bmatrix}X_1(z) & 0\\ 0 & X_2(z)\end{bmatrix}$, $X_j(z)=\prod\limits_{k=1}^{n}(z-a_k)^{-\frac{1}{2}+i\beta_j}(z-b_k)^{-\frac{1}{2}-i\beta_j}$, $\beta_j=\dfrac{\ln|M_j|}{2\pi}$, $j=1,2$, $\Delta_2M_j=4\Delta_0\Delta_1-(A_1+A_4)+(-1)^j\sqrt{(A_1+A_4)^2-16\Delta_0\Delta_1}<0$, $j=1,2$, $P_{n-1}(z)=(P_1,P_2)^T$, $P_j=\sum_{q=0}^{n-1}C_q^{(j)}z^{n-1-q}$, $j=1,2$.

To define $C_q^{(j)}$, $j=1,2$, $q=\overline{0,n-1}$, we use that value of the principal vector on the L_k' is equal to P_k+iQ_k, $k=\overline{1,n}$.

If suppose $\overset{(2)}{\tau}=-A_0$ and $\overset{(1)}{\eta}=B_0$, then we can write [3].

$$\int_{L_k'}[A_0(t_0)+iB_0(t_0)]dt_0=\theta_k-iP_k,\quad k=\overline{1,n}. \tag{14}$$

Now note that on the basis of (11)-(13) the elementary calculations gives

$$A_0(t_0)+iB_0(t_0)=f'(t_0)+\frac{1}{y_2-y_1}\begin{bmatrix}y_2 & -1\\ -y_1 & 1\end{bmatrix}R(t_0)-2\phi(t_0). \tag{15}$$

Substituting in formula (14) the value $A_0(t_0)+iB_0(t_0)$ appearing in (15) we obtain the system of linear equations for determing $C_q^{(j)}$, $j=1,2$, $q=\overline{0,n-1}$.

Having found $C_q^{(j)}$, $j=1,2$, $q=\overline{0,n-1}$, we can be define $\phi(z)$, hence $\Psi(z)$, $\varphi(z)$ and $\psi(z)$.

Finally for (9) we obtain uniquely solution of the basic mixed problem.

R E F E R E N C E S

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