Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 24, 2010

SOLUTION OF ONE SPECIFIC EQUATION CONNECTED WITH PROBLEMS OF THE LIGHT SCATTERING THEORY

Shulaia D., Sharashidze N.

Abstract. The aim of this paper is to study in the class of Hölder functions characteristic integral equation of the light scattering theory. Using the theory of singular integral equations, necessary and sufficient conditions for the solvability corresponding this nonhomogeneous equation are given. A theorem on the expansion of the solution of this equation in terms of eigenfunctions of discrete and continuous spectra of the characteristic equation is represented.

Keywords and phrases: Characteristic equation, eigenfunctions, singular integral.

AMS subject classification: 45B05, 45E05.

Consider the well known integro-differential equation of the light scattering theory [1]

$$\mu\frac{\partial I(\tau,\mu,x)}{\partial\tau}=(\alpha(x)+\beta)I(\tau,\mu,x)$$
$$-\frac{\lambda}{2}A\alpha(x)\int_{-\infty}^{+\infty}\alpha(x')\int_{-1}^{+1}I(\tau,\mu',x')d\mu'dx',$$
$$\tau\in(-\infty,+\infty),\quad \mu\in(-1,+1),\quad x\in(-\infty,+\infty),$$

where $I(\tau,\mu,x)$ is the angular density, τ the optical depth, μ is the cosine of the angle with respect to the τ- axis, x is so-called dimensionless frequency, λ is the fatigue probability in an elementary scattering, $\alpha(x)>0$ is so-called profile coefficient of absorption, A is the normalizing constant

$$A\int_{-\infty}^{+\infty}\alpha(x)dx=1,$$

coefficient $\beta>0$ characterized also the absorption.

By using the Fourier transformation with respect to spatial variable τ it is possible the considered equation reduce to the following form integral equation

$$(t(\alpha(x)+\beta)-\mu)\psi_t(\mu,x) \tag{1}$$
$$=t\frac{\lambda}{2}A\alpha(x)\int_{-\infty}^{+\infty}\alpha(x')\int_{-1}^{+1}\psi_t(\mu',x')d\mu'dx'+f(\mu,x),$$
$$\mu(-1,+1),\quad x\in(-\infty,+\infty)$$

(see e.g. [2,3]). Here t is a parameter and the real valued function $f(\mu,x)$ is defined from boundary conditions of the considered problem. We shall study the problem where $f(\mu, x)$ is a continuous function satisfying H condition with respect to μ and integrable with respect to x. The corresponding homogeneous equation has the form

$$(\nu(\alpha(x)+\beta)-\mu)\varphi_\nu(\mu,x)=\nu\frac{\lambda}{2}A\alpha(x)\int_{-\infty}^{+\infty}\alpha(x')\int_{-1}^{+1}\varphi_(\nu)(\mu',x')d\mu'dx'). \qquad (2)$$

Here ν is a parameter, the values of ν for which (2) has nonzero solutions are the eigenvalues of the equation.

We can prove the result concerning the eigenvalues of homogeneous equation

Theorem 1. *The discrete spectrum of (2) not belongs in the segment* $[-1/\beta, 1/\beta]$.

Theorem 2. *If*

$$\frac{\lambda}{2}A\int_{-\infty}^{+\infty}\frac{\alpha^2(x)}{\alpha(x)+\beta}dx>1,$$

then the discrete spectrum of (2) consist of the two imaginary points $\{\pm\nu_0\}$ *which corresponds to the two eigenfunctions, when*

$$\frac{\lambda}{2}A\int_{-\infty}^{+\infty}\frac{\alpha^2(x)}{\alpha(x)+\beta}dx<1$$

and

$$\frac{\lambda}{2}A\int_{-\infty}^{+\infty}\int_{-1}^{+1}\frac{\alpha^2(x)}{\alpha(x)+\beta-\mu\beta}d\mu dx>1,$$

then the discrete spectrum of (2) consist of two real points $\{\pm\nu_0\}$, *but if the last condition not is fulfil then the set of discrete spectrum is empty.*

Theorem 3. *Let* $\nu\in(-\frac{1}{\beta},\frac{1}{\beta})$. *The function*

$$\varphi_{\nu,(\xi)}(\mu,x)=\frac{\nu\lambda A}{2}\rho(\nu)\frac{\alpha(x)\alpha(\xi)}{\nu(\alpha(x)+\beta)-\mu}$$

$$+(\delta(\xi-x)-\frac{\nu\lambda A}{2}\int_{-1}^{+1}\rho(\nu)\frac{\alpha(x)\alpha(\xi)}{\nu(\alpha(x)+\beta)-\mu'}d\mu')\delta(\nu(\alpha(x)+\beta)-\mu)), \qquad (3)$$

$$\mu\in(-1,+1),\quad x\in(-\infty,+\infty),$$

where $\xi\in\boldsymbol{M}(\nu)$, δ *is the Dirac function,*

$$\rho^{-1}(\nu)=1-\frac{\nu\lambda A}{2}\int_{\boldsymbol{N}(\nu)}\int_{-1}^{+1}\frac{\nu\alpha^2(x)}{\nu(\alpha(x)+\beta)-\mu}d\mu dx,$$

$$\boldsymbol{M}(\nu)=\{x\in(-\infty,+\infty):\mid\nu\mid(\alpha(x)+\beta)<1\}$$

and

$$\boldsymbol{N}(\nu)=\{x\in(-\infty,+\infty):\mid\nu\mid(\alpha(x)+\beta)>1\},$$

is the singular eigenfunction (in K. Case sense [3]) of the characteristic equation (2).

The eigenfunctions $\varphi_\nu(\mu,x)$ *obey the following orthogonality condition*

$$\int_{-\infty}^{+\infty}\int_{-1}^{+1}\varphi_\nu(\mu,x)\varphi_{\nu'}(\mu,x)d\mu dx=0,$$

where $\nu \neq \nu'$.

We have to construct the solution of equation (1). We can prove

Theorem 4. *If* $t \bar{\in} \{\pm\nu_0\} \bigcup [-\frac{1}{\beta}, \frac{1}{\beta}]$, *then the equation (1) has unique solution which can represent in the form*

$$\psi_t(\mu, x) = c_{+\nu_0}(t)\varphi_{+\nu_0}(\mu, x) + c_{-\nu_0}(t)\varphi_{-\nu_0}(\mu, x) + \int_{-1/\beta}^{1/\beta} \int_{\boldsymbol{M}(\nu)} \varphi_{\nu,(\xi)}(\mu, x) u(t, \mu, \xi) d\xi d\nu$$

where $c_{\pm\nu_0}$ *and* $u(t, \mu, \xi)$ *defined explicitly.*

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Received 13.06.2010; revised 15.10.2010; accepted 12.11.2010.

Authors' addresses:

D. Shulaia
I. Vekua Institute of Applied Mathematics of
Iv. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia
E-mail: dazshul@yahoo.com

N. Sharashidze
Iv. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia